Systems Analysis and Control

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Lecture 3: Linearization

Introduction

In this Lecture, you will learn:

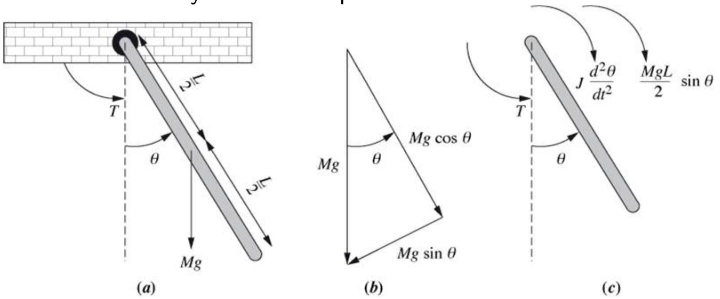
How to Linearize a Nonlinear System *System*.

- Taylor Series Expansion
- Derivatives
- L'hoptial's rule
- Multiple Inputs/ Multiple States

Lets **Start** with an Example

A Simple Pendulum

Consider the rotational dynamics of a pendulum:



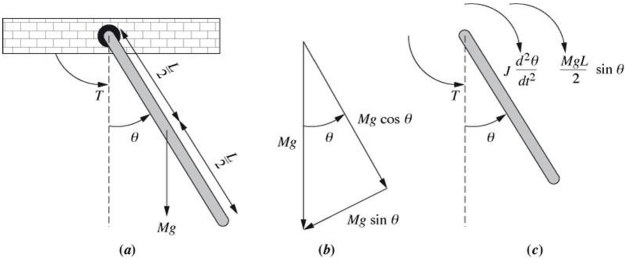
- The *input* is a motor-driven moment, T.
- The *output* is the angle, θ.
- The moment of inertia about the pivot point is J.
- The only external force is gravity, Mg, applied at the center of mass.
- Force creates a moment about the pivot (See Figure b)):

$$N = -Mg\sin\theta \cdot \frac{l}{2}$$

A Simple Pendulum

The governing equation is Newton's law:

$$\ddot{\theta} = \frac{N}{J}$$



Equations of Motion (EOM):

$$\ddot{\theta} = -\frac{Mgl}{2J}\sin\theta + \frac{T}{J}$$

$$y = \theta$$

First-order form: Let $x_1 = \theta$, $x_2 = \dot{\theta}$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{Mgl}{2J}\sin x_1 + \frac{T}{J}$$

$$y = x_1$$

First-order form:

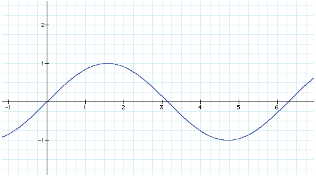
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{Mgl}{2J}\sin x_1 + \frac{T}{J}$$
$$y = x_1$$

Although we have the system in first-order form, it cannot be put in state-space because of the $\sin x_1$ term.

What to do???

Although $\sin x$ is nonlinear, small sections look linear.

- **Near** $x = 0$**:** $\sin x \cong x$
- **Near** $x = \pi/2$**:** $\sin x \cong 1$
- **Near** $x = \pi$**:** $\sin x \cong \pi - x$



We must use these linear approximations *very carefully*!

Accuracy of the Small Angle Approximation

The approximation will only be accurate for a narrow band of x.

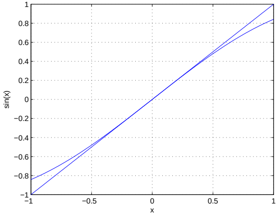


Figure: $\sin(x)$ and x near $x_0 = 0$

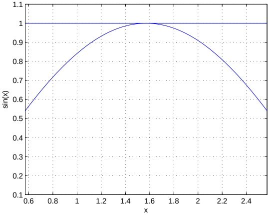


Figure: $\sin(x)$ and x near $x_0 = \frac{\pi}{2}$

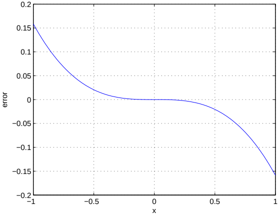


Figure: Error near $x = 0$

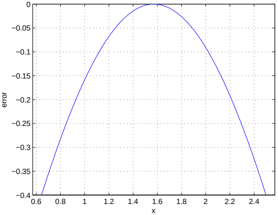


Figure: Error near $x = \frac{\pi}{2}$

- **80% Accuracy:** $x \in [-1.2, 1.2]$
- **95% Accuracy:** $x \in [-.7, .7]$
- **80% Accuracy:** $x \in [.9, 2.2]$
- **95% Accuracy:** $x \in [1.25, 1.9]$

Linear Approximation

We can use the tangent to approximate a nonlinear function near a point x_0.

Key Point: The approximation is *tangent* to the function at the point x_0.

$$f(x) \cong ax + b$$

- The slope is given by

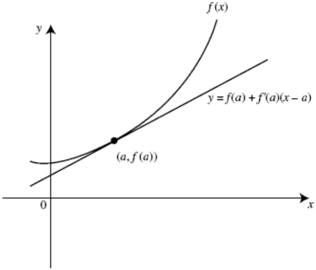
$$a = \frac{d}{dx} f(x)|_{x=x_0}$$

- The *y-intercept* is given by

$$b = f(x_0) - ax_0$$

The **linear approximation** is given by

$$f(x) \cong f(x_0) + \frac{d}{dx} f(x)|_{x=x_0} (x - x_0)$$



A General Method For Linear Approximation

Problem: Approximate the scalar function $f(x)$ near the point x_0 using

$$y(x) = ax + b$$

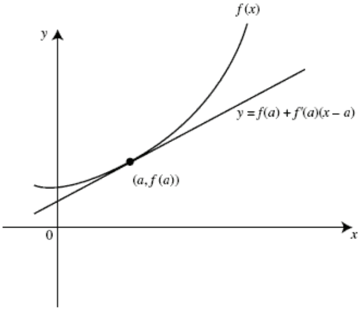


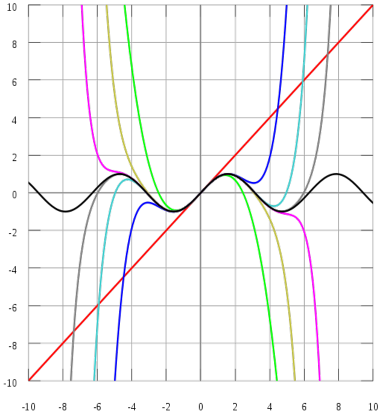
Figure 9.2-1

The **Linear Approximation** is given by

$$y(x) = f(x_0) + \frac{d}{dx} f(x)|_{x=x_0}(x - x_0)$$

Linear Approximation

Note: The Linear Approximation is just the first two terms in the Taylor Series representation.



$$f(x) = f(x_0) + \frac{d}{dx} f(x)|_{x=x_0} \frac{(x - x_0)}{1!} + \frac{d}{dx} f(x)|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots$$

Example: Pendulum

Return to the dynamics of a pendulum:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{Mgl}{2J}\sin x_1 + \frac{1}{J}T$$
$$y = x_1$$

The nonlinear term is $\sin x_1$

- We want to linearize $\sin x_1$.
- Choose an operating point, x_0!
  - Depends on what we want to do!
  - Options are limited.

Disturbance rejection: $x_0 = 0$

Balance: $x_0 = \pi$

Tracking: $x_0 = ???$

Example: Balance an Inverted Pendulum

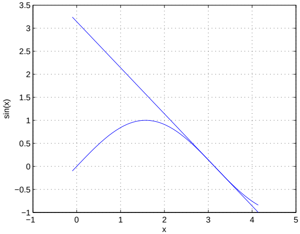
Applications: Walking robots.

An inverted pendulum has $x \cong \pi$.

- *Tangent:*
$$a = \frac{d}{dx} f(x)|_{x=x_0} = \cos(\pi) = -1$$
- *Intersect:*
$$b = f(x_0) - a x_0 = \sin(\pi) + \pi = \pi.$$
- $f(x_0) = \sin(\pi) = 0$
- Finally, for $x \cong \pi$
$$\sin(x) \cong \pi - x$$

This gives the first-order dynamics:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{Mgl}{2J} x_1 - \frac{Mgl}{2J}\pi + \frac{1}{J} T$$
$$y = x_1$$

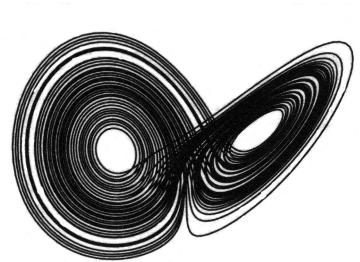


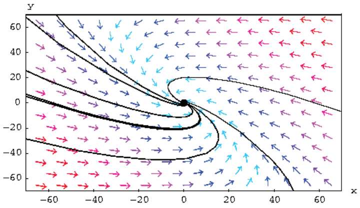
New Problem: The constant term $-\frac{Mgl}{2J}\pi$ doesn't fit in state-space:

$$\dot{x} = Ax + Bu$$

Equilibrium Points

Problem: $\dot{x} \neq 0$ when $x = 0$. We need a new concept





Definition 1.

x_0 is an **Equilibrium Point** of $\dot{x} = f(x)$ if $\dot{x} = 0$ when $x = x_0$. i.e. $f(x_0) = 0$

- Nonlinear systems may have *many* equilibrium points.
- Linear (affine) systems only have one equilibrium point.
- In a state-space system, $x_0 = 0$ is the *unique* equilibrium point.

For state-space, we need $x_0 = 0$ to be the equilibrium point.
The nonlinear pendulum has infinitely many equilibria.

- Down equilibria: $x_0 = 0 + 2\pi n$ for $n = 1, \cdots, \infty$
- Up equilibria: $x_0 = \pi + 2\pi n$ for $n = 1, \cdots, \infty$

Our linearized pendulum has one equilibrium at $x_0 = \pi$:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{Mgl}{2J}(x_1 - \pi) + \frac{1}{J}T, \qquad y = x_1$$

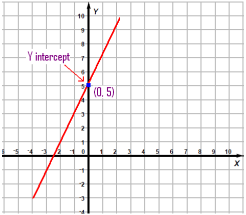
Problem: For state-space (or any standard form), we require $x_0 = 0$.

Solution: Define a new variable $\Delta x = x - x_0$

- Then

$$\Delta\dot{x} = \dot{x} = a(\Delta x - \frac{b}{a}) + b = a\Delta x$$

- Thus $\Delta x_0 = 0$ is the equilibrium!!!



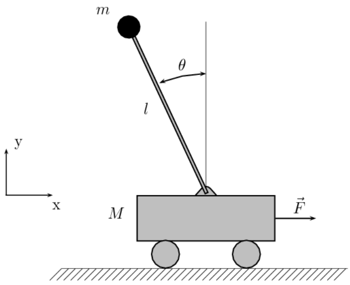
Measuring Displacement from Equilibrium

Pendulum Example

Return to the pendulum.

- Equilibrium at $x_{0,1} = \pi$, $x_{0,2} = 0$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{Mgl}{2J}(x_1 - \pi) + \frac{1}{J}T$$



- Let

$$\Delta x_1 = x_1 - \pi$$
$$\Delta x_2 = x_2$$

- New Dynamics:

$$\Delta \dot{x}_1 = \Delta x_2$$
$$\Delta \dot{x}_2 = \frac{Mgl}{2J}\Delta x_1 + \frac{1}{J}T$$

Δx_1 is angle from the vertical.

Measuring Displacement from Equilibrium

Now we are ready for state-space.

New Dynamics:

$$\Delta\dot{x}_1 = \Delta x_2$$

$$\Delta\dot{x}_2 = \frac{Mgl}{2J}\Delta x_1 + \frac{1}{J}T$$

State-Space Form:

$$\Delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{2J} & 0 \end{bmatrix}\Delta x + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\Delta x$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{2J} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

Although not for the pendulum, you may sometimes need to linearize functions of the input and output!

Example: Balance an Inverted Pendulum

Applications: Walking robots.

Example: Balance an Inverted Pendulum

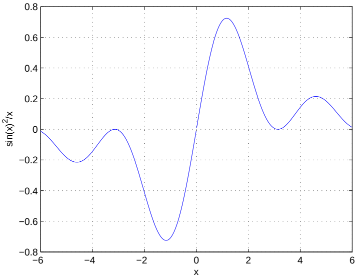
Applications: Segway.

Numerical Example: Using l'Hôpital's rule

Occasionally you will encounter a system such as

$$\ddot{x}(t) = -\dot{x}(t) + \frac{\sin^2(x(t))}{x(t)}$$

where you want to linearize about the **zero equilibrium**.



The nonlinear term is $\frac{\sin^2 x}{x}$ with equilibrium point $x_0 = 0$. To linearize this term about $x_0 = 0$, use the formula:

$$f(x) \cong f(x_0) + f'(x_0)(x - x_0)$$

To do this we must calculate $f(x_0)$ and $f'(x_0)$.

Lets start with $f(x_0)$. Initially, we see that $f(0) = \frac{0}{0}$, which is indeterminate. To help, we use L'hopital's Rule.

L'hopital's Rule

Theorem 2 (l'Hôpital's Rule).

If $g(0) = 0$ *and* $h(0) = 0$*, then*

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{g'(x)}{h'(x)}$$

If we apply this to $f(x) = \dfrac{\sin^2(x)}{x}$, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2\sin x \cos x}{1} = \frac{0}{1} = 0$$

which is as expected. Now,

$$f'(x) = \frac{2\sin x \cos x}{x} - \frac{\sin^2 x}{x^2} = \frac{2x\sin x \cos x - \sin^2 x}{x^2}$$

As before,

$$f'(0) = \frac{0}{0}$$

Example Continued

So once more we apply L'hopital's rule:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\sin x\cos x + 2x\cos^2 x - 2x\sin^2 x - 2\sin x\cos x}{2x}$$
$$= \lim_{x\to 0} \frac{(2x(\cos^2 x + -\sin^2 x)}{2x} = \frac{0}{0}$$

Ooops, we must apply l'Hôpital's rule AGAIN:

$$\lim_{x\to 0} \frac{(2x(\cos^2 x + -\sin^2 x)}{2x}$$
$$= \frac{2(\cos^x - \sin^2) - 8x\cos x\sin x}{2} = \frac{2}{2} = 1$$

Which was a lot of work for such a simple answer (easier way?). We have the linearized equation of motion:

$$\ddot{x}(t) = -\dot{x}(t) + 1\cdot x(1) + 0$$

Which in standard form is $x_1 = x$, $x_2 = \dot{x}$, so

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x$$

Summary

What have we learned today?

How to Linearize a Nonlinear System *System*.

- Taylor Series Expansion
- Derivatives
- L'hoptial's rule
- Multiple Inputs/ Multiple States

Next Lecture: Laplace Transform