# Systems Analysis and Control

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Lecture 6: Calculating the Transfer Function

# Introduction

In this Lecture, you will learn: Transfer Functions

- Transfer Function Representation of a System
- State-Space to Transfer Function
- Direct Calculation of Transfer Functions

Block Diagram Algebra

- Modeling in the Frequency Domain
- Reducing Block Diagrams

# Previously:

## The Laplace Transform of a Signal

**Definition:** We defined the Laplace transform of a **Signal**.

- **Input**, $\hat{u} = \Lambda u$.
- **Output**, $\hat{y} = \Lambda y$

### Theorem 1.

*For a bounded, linear, causal, time-invariant system, $y = Gu$, there exists a* **Transfer Function**, $\hat{G}$, *so that the ratio of input to output is*

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \hat{G}(s)$$

In this lecture, we will discuss several ways of finding the *Transfer Function*.

**State-Space:**

$$\dot{x}(t) = -x(t) + u(t)$$
$$y(t) = x(t) - .5u(t) \qquad x(0) = 0$$

Apply the Laplace transform to the first equation:

$$\Lambda\bigg(\dot{x}(t) = -x(t) + u(t)\bigg) \qquad \text{which gives} \qquad s\hat{x}(s) + x(0) \quad = -\hat{x}(s) + \hat{u}(s).$$

Noting that $x(0) = 0$ and solving for $\hat{x}(s)$ gives

$$(s+1)\hat{x}(s) = \hat{u}(s) \qquad \text{and so} \qquad \hat{x}(s) = \frac{1}{s+1}\hat{u}(s).$$

Similarly, the second equation gives $\hat{y}(s)$:

$$\hat{y}(s) = \hat{x}(s) - .5\hat{u}(s) = \frac{1}{s+1}\hat{u}(s) - .5\hat{u}(s) = \frac{1 - .5(s+1)}{s+1}\hat{u}(s) = \frac{1}{2}\frac{s-1}{s+1}\hat{u}(s)$$

Thus we have the Transfer Function:

$$\hat{G}(s) = \frac{1}{2}\frac{s-1}{s+1}$$

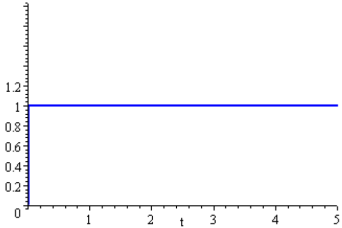
# Transfer Functions

## Example: Step Response

The *Transfer Function* provides a convenient way to find the response to inputs.

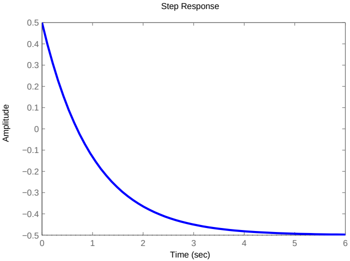
**Step Input Response:** $\hat{u}(s) = \frac{1}{s}$

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{2}\frac{s-1}{s+1}\frac{1}{s} = \frac{1}{2}\frac{s-1}{s^2+s}$$
$$= \frac{1}{2}\left(\frac{2}{s+1} - \frac{1}{s}\right)$$




Consulting our table of Laplace Transforms,

$$y(t) = \frac{1}{2}\Lambda^{-1}\frac{2}{s+1} - \frac{1}{2}\Lambda^{-1}\frac{1}{s}$$
$$= e^{-t} - \frac{1}{2}\mathbf{1}(t)$$



# Transfer Functions

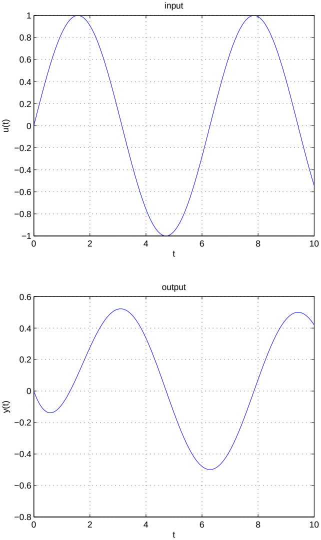
## Example: Sinusoid Response

**Sine Function:** $\hat{u}(s) = \frac{1}{s^2+1}$

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{2}\frac{s-1}{s+1}\frac{1}{s^2+1}$$
$$= \frac{1}{2}\frac{s-1}{s^3+s^2+s+1}$$
$$= \frac{1}{2}\left(\frac{s}{s^2+1} - \frac{1}{s+1}\right)$$

Consulting our table of Laplace Transforms,

$$y(t) = \frac{1}{2}\cos t - \frac{1}{2}e^{-t}$$



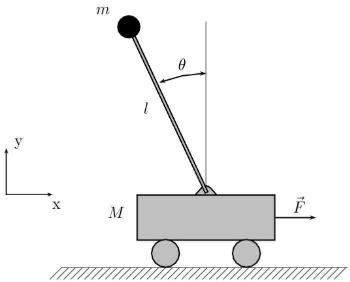

Note that this is the same answer we got by integration in Lecture 4.

# Inverted Pendulum Example

Return to the pendulum.

**Dynamics:**

$$\ddot{\theta}(t) = \frac{Mgl}{2J}\theta(t) + \frac{1}{J}T(t)$$
$$y(t) = \theta(t)$$




For the first equation,

$$s^2\hat{\theta}(s) - \dot{\theta}(0) - s\theta(0) = \frac{Mgl}{2J}\hat{\theta}(s) + \frac{1}{J}\hat{T}(s)$$

Set $\dot{x}(0) = 0$ and $x(0) = 0$ and solve for $\hat{\theta}(s)$:

$$\hat{\theta}(s) = \frac{1}{J}\frac{1}{s^2 - \frac{Mgl}{2J}}\hat{T}(s)$$

**Second Equation:** $\hat{y}(s) = \hat{\theta}(s)$

**Transfer Function:**

$$\hat{G}(s) = \frac{1}{J}\frac{1}{s^2 - \frac{Mgl}{2J}}$$

# Inverted Pendulum Example: Impulse Response

**Impulse Input:** $\hat{u}(s) = 1$

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{J}\frac{1}{s^2 - \frac{Mgl}{2J}}$$

$$= \frac{1}{J}\frac{1}{(s - \sqrt{\frac{Mgl}{2J}})(s + \sqrt{\frac{Mgl}{2J}})}$$

$$= \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(\frac{1}{s - \sqrt{\frac{Mgl}{2J}}} - \frac{1}{s + \sqrt{\frac{Mgl}{2J}}}\right)$$

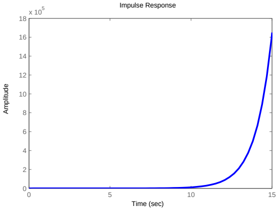


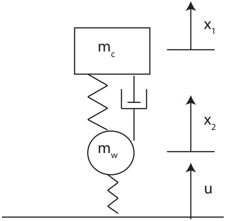

Figure: Impulse Response with $g = l = J = 1$, $M = 2$

In time-domain:

$$y(t) = \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(e^{\sqrt{\frac{Mgl}{2J}}t} - e^{-\sqrt{\frac{Mgl}{2J}}t}\right)$$

Pendulum Accelerates to infinity!

# Constructing the Transfer Function: Suspension System



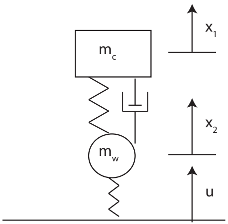

Recall the dynamics:

$$\ddot{z}_1(t) = -\frac{K_1}{m_c} z_1(t) - \frac{c}{m_c}\dot{z}_1(t) + \frac{K_1}{m_c} z_2(t) + \frac{c}{m_c}\dot{z}_2(t)$$

$$\ddot{z}_4(t) = \frac{K_1}{m_w} z_1(t) + \frac{c}{m_w}\dot{z}_1(t) - \left(\frac{K_1}{m_w} + \frac{K_2}{m_w}\right) z_2(t) - \frac{c}{m_w}\dot{z}_2(t) - \frac{K_2}{m_w} u(t)$$

$$y(t) = \left[z_2(t)\right]$$

# Constructing the Transfer Function: Suspension System




Apply the Laplace Transform to the dynamics:

$$s^2\hat{z}_1(s) = -\frac{K_1}{m_c}\hat{z}_1(s) - \frac{c}{m_c}s\hat{z}_1(s) + \frac{K_1}{m_c}\hat{z}_2(s) + \frac{c}{m_c}s\hat{z}_2(s)$$

$$s^2\hat{z}_2(s) = \frac{K_1}{m_w}\hat{z}_1(s) + \frac{c}{m_w}s\hat{z}_1(s) - \left(\frac{K_1}{m_w} + \frac{K_2}{m_w}\right)\hat{z}_2(s) - \frac{c}{m_w}s\hat{z}_2(s) - \frac{K_2}{m_w}\hat{u}(s)$$

$$\hat{y}(s) = \hat{z}_2(s)$$

# Constructing the Transfer Function: Suspension System

We isolate the $z_1$ and $z_2$ terms:

$$\left(s^2 + \frac{c}{m_c}s + \frac{K_1}{m_c}\right)\hat{z}_1(s) = \left(\frac{K_1}{m_c} + \frac{c}{m_c}s\right)\hat{z}_2(s)$$
$$\left(s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}\right)\hat{z}_2(s) = \left(\frac{K_1}{m_w} + \frac{c}{m_w}s\right)\hat{z}_1(s) - \frac{K_2}{m_w}\hat{u}(s)$$
$$\hat{y}(s) = \hat{z}_2(s)$$

Which yields

$$\hat{z}_1(s) = \frac{\left(\frac{K_1}{m_c} + \frac{c}{m_c}s\right)}{\left(s^2 + \frac{c}{m_c}s + \frac{K_1}{m_c}\right)}\hat{z}_2(s)$$

$$\hat{z}_2(s) = \frac{\frac{K_1}{m_w} + \frac{c}{m_w}s}{s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}}\hat{z}_1(s) - \frac{\frac{K_2}{m_w}}{s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}}\hat{u}(s)$$

# Constructing the Transfer Function: Suspension System

Now we can plug in for $\hat{z}_1$ and solve for $\hat{z}_2$:

$$\hat{z}_2(s) =$$

$$\frac{K_2(m_c s^2 + cs + K_1)}{m_c m_w s^4 + c(m_w + m_c)s^3 + (K_1 m_c + K_1 m_w + K_2 m_c)s^2 + cK_2 s + K_1 K_2}\hat{u}(s)$$

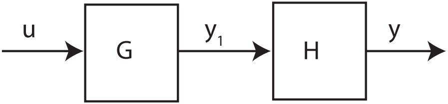
Compare to the State-Space Representation:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{m_c} & -\frac{c}{m_c} & \frac{K_1}{m_c} & \frac{c}{m_c} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{m_w} & \frac{c}{m_w} & -\left(\frac{K_1}{m_w} + \frac{K_2}{m_w}\right) & -\frac{c}{m_w} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{K_2}{m_w} \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}u(t)$$

**Note:** We only used one output to find the transfer function.

The interconnection of systems can be represent by block diagrams.




**Cascade of Systems:** Suppose we have two systems: $G$ and $H$.

## Definition 2.

The **Cascade** or **Series** interconnection of two systems is

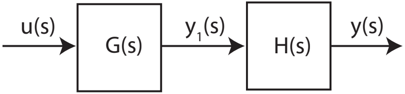
$$y_1 = Gu \qquad y = Hy_1$$

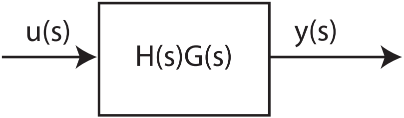
or

$$y = H(G(u))$$

# Block Diagrams

## Series Connection




The Transfer function of a Series interconnection is Simple

- The output of system 1 is the input to system 2.
- Let $\hat{G}(s)$ and $\hat{H}(s)$ be the transfer functions for $G$ and $H$.
- Apply the Laplace transform to get

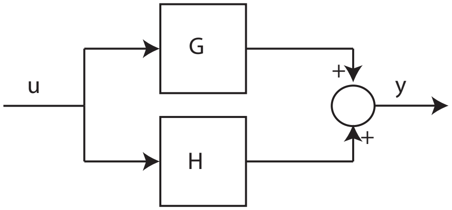
$$\hat{y}_1(s) = \hat{G}_1(s)\hat{u}(s) \qquad \hat{y}(s) = \hat{H}(s)\hat{y}_1(s) = \hat{H}(s)\hat{G}(s)\hat{u}(s)$$

- The *Transfer Function*, $\hat{T}(s)$ for the cascade of $G$ and $H$ is

$$\hat{T}(s) = \hat{H}(s)\hat{G}(s)$$

**Note:** The order of the Transfer Functions!

The parallel Interconnection is even simpler.




**Parallel Interconnection:** Suppose we have two systems: $G$ and $H$.

## Definition 3.

The **Parallel** interconnection of two systems is

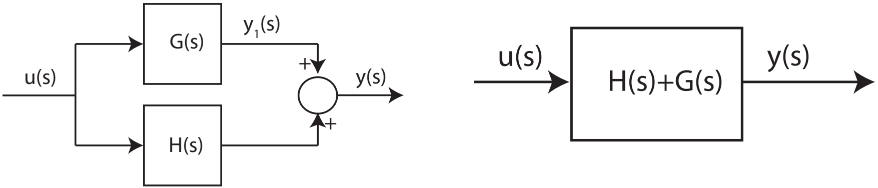
$$y_1 = Gu \qquad y_2 = Hu \qquad y = y_1 + y_2$$

or

$$y = H(u) + G(u)$$

# Block Diagrams

## Parallel Connection



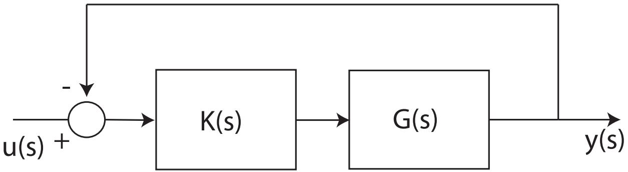

The Transfer function of a Parallel interconnection is trivial

- Apply the Laplace transform to get

$$\hat{y}(s) = \hat{y}_1(s) + \hat{y}_2(s) = \hat{G}(s)\hat{u}(s) + \hat{H}(s)\hat{u}(s) = \left(\hat{H}(s) + \hat{G}(s)\right)\hat{u}(s)$$

- The *Transfer Function*, $\hat{T}(s)$ for the parallel interconnection of $G$ and $H$ is

$$\hat{T}(s) = \hat{H}(s) + \hat{G}(s)$$




**Feedback:**

- **Controller:** $z = K(u - y)$ **Plant:** $y = Gz$

Applying the Laplace Transform gives

$$\hat{z}(s) = -\hat{K}(s)\hat{y}(s) + \hat{K}(s)\hat{u}(s) \qquad \hat{y}(s) = \hat{G}(s)\hat{u}_i(s)$$

so

$$\hat{y}(s) = \hat{G}(s)\hat{z}(s) = -\hat{G}(s)\hat{K}(s)\hat{y}(s) + \hat{G}(s)\hat{K}(s)\hat{u}(s)$$

Solving for $\hat{y}(s)$,

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$

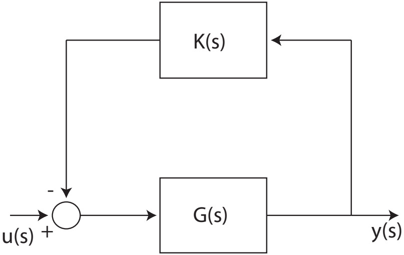
# Block Diagrams

## Upper Feedback Interconnection

There is an alternative Feedback interconnection

- Let $u$ be the external input/disturbance
- $y$ is the output

$$\hat{y}(s) = \hat{G}(s)\hat{z}(s)$$
$$\hat{z}(s) = u(s) - \hat{K}(s)\hat{y}(s)$$




Which yields

$$\hat{y}(s) = \hat{G}(s)\left(u(s) - \hat{K}(s)\hat{y}(s)\right) = \hat{G}(s)\hat{u}(s) - \hat{G}(s)\hat{K}(s)\hat{y}(s)$$

hence the Transfer Function is given by

$$\hat{y}(s) = \frac{\hat{G}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s).$$

# The Effect of Feedback: Impulse Response



**Transfer Function**

$$\hat{G}(s) = \frac{1}{Js^2 - \frac{Mgl}{2}}$$

**Controller:** Static Gain: $\hat{K}(s) = K$
**Input:** Impulse: $\hat{u}(s) = 1$.
**Closed Loop:** Lower Feedback

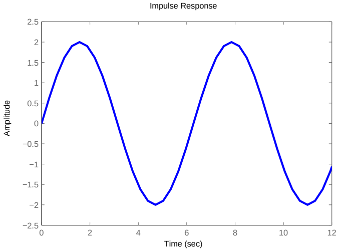
$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s) = \frac{\frac{K}{Js^2 - \frac{Mgl}{2}}}{1 + \frac{K}{Js^2 - \frac{Mgl}{2}}} = \frac{K}{Js^2 - \frac{Mgl}{2} + K}$$

**First Case:**

- If $K > \frac{Mgl}{2}$, then $K - \frac{Mgl}{2} > 0$, so

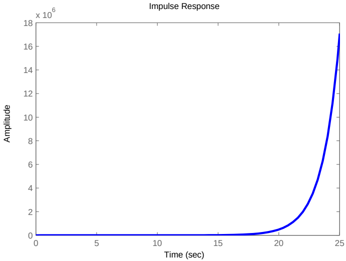
$$\hat{y}(s) = \frac{K/J}{s^2 + \left(K/J - \frac{Mgl}{2J}\right)}$$

$$y(t) = \frac{K}{J\sqrt{K/J - \frac{Mgl}{2J}}} \sin\left(\sqrt{K/J - \frac{Mgl}{2J}}t\right)$$



# The Effect of Feedback: Impulse Response

## Inverted Pendulum Model




**Second Case:**

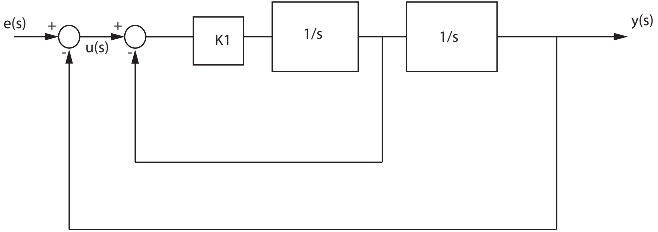
- If $K < \frac{Mgl}{2}$, then $K - \frac{Mgl}{2} < 0$, so

$$\hat{y}(s) = \frac{K}{J}\left(\frac{1}{s - \sqrt{K/J - \frac{Mgl}{2J}}} + \frac{1}{s + \sqrt{K/J - \frac{Mgl}{2J}}}\right)$$

$$y(t) = \frac{K}{J}\left(e^{\sqrt{K/J - \frac{Mgl}{2J}}t} + e^{-\sqrt{K/J - \frac{Mgl}{2J}}t}\right)$$

Important: Value of $K$ determines stability vs. instability

Now lets look at how to reduce a more complicated interconnections




Label

- The output from the inner loop $z$
- The input to the inner loop $u$

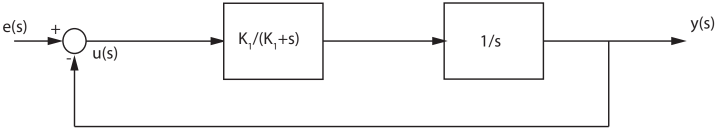
First **Close the Inner Loop** using the *Lower Feedback Interconnection*.

$$\hat{z}(s) = \frac{\frac{K_1}{s}}{\frac{K_1}{s}+1}\hat{u}(s) = \frac{K_1}{K_1+s}\hat{u}(s)$$

# Block Diagrams

## Reduction

We now have a reduced Block Diagram




Again, apply the *Lower Feedback Interconnetion*:

$$\hat{y}(s) = \frac{\frac{K_1}{s(K_1+s)}}{1 + \frac{K_1}{s(K_1+s)}}\hat{e}(s) = \frac{K_1}{s(K_1+s)+K_1}\hat{e}(s)$$

So the Transfer function is $\hat{T}(s) = \dfrac{K_1}{s^2 + K_1 s + K_1}$

# Summary

What have we learned today?

Transfer Functions

- Transfer Function Representation of a System
- State-Space to Transfer Function
- Direct Calculation of Transfer Functions

Block Diagram Algebra

- Modeling in the Frequency Domain
- Reducing Block Diagrams

**Next Lecture: Partial Fraction Expansion**