Systems Analysis and Control

Matthew M. Peet
Illinois Institute of Technology

Lecture 8: Response Characteristics

Overview

In this Lecture, you will learn:

Characteristics of the Response

- Stability

Real Poles

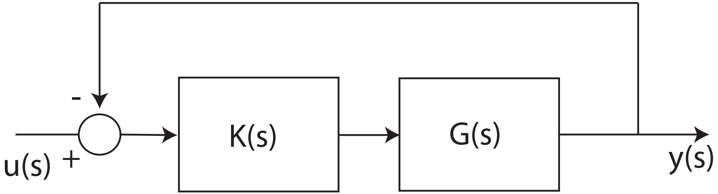
- Steady-State Error
- Rise Time
- Settling Time

Complex Poles

- Complex Pole Locations
- Damped/Natural Frequency
- Damping and Damping Ratio

Feedback Control

Recall the Feedback Interconnection



Feedback:

- **Controller:** $u_i = K(u - y)$
- **Plant:** $y = Gu_i$

The output signal is $\hat{y}(s)$,

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$

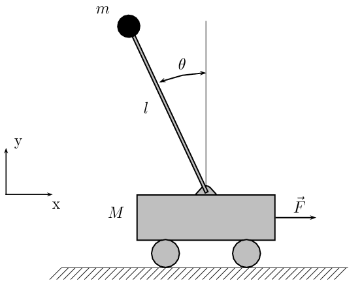
Controlling the Inverted Pendulum Model

Open Loop Transfer Function

$$\hat{G}(s) = \frac{1}{Js^2 - \frac{Mgl}{2}}$$

Controller: Static Gain: $\hat{K}(s) = K$

Input: Impulse: $\hat{u}(s) = 1$.



Closed Loop: Lower Feedback

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)}\hat{u}(s) = \frac{\frac{K}{Js^2-\frac{Mgl}{2}}}{1+\frac{K}{Js^2-\frac{Mgl}{2}}} = \frac{K}{Js^2 - \frac{Mgl}{2} + K}$$

Controlling the Inverted Pendulum Model

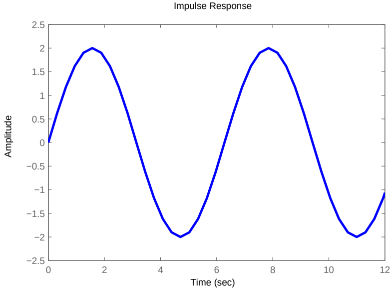
Closed Loop Impulse Response:

Lower Feedback

$$\hat{y}(s) = \frac{K}{Js^2 - \frac{Mgl}{2} + K}$$

Traits:

- Infinite Oscillations
- Oscillates about 0.

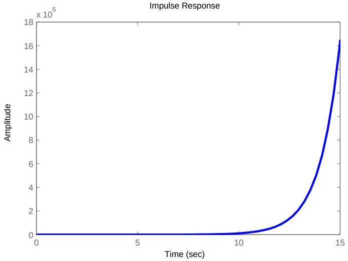


Open Loop Impulse Response:

$$\hat{y}(s)$$

$$= \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(\frac{1}{s-\sqrt{\frac{Mgl}{2J}}} - \frac{1}{s+\sqrt{\frac{Mgl}{2J}}}\right)$$

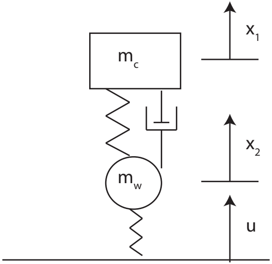
Unstable!



Controlling the Suspension System

Open Loop Transfer Function:
Set $m_c = m_w = g = c = K_1 = K_2 = 1$.

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}$$



Controller: Static Gain: $\hat{K}(s) = k$

Closed Loop: Lower Feedback

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$
$$= \frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3 + k)s^2 + (1 + k)s + (1 + k)}$$

Closed Loop Step Response:

$$\hat{y}(s) = \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}\frac{1}{s}$$

High k:

- Overshot the target
- Quick Response
- Closer to desired value of f

Low k:

- Slow Response
- No overshoot
- Final value is farther from 1.

Questions:

- Which Traits are important?
- How to predict the behaviour?

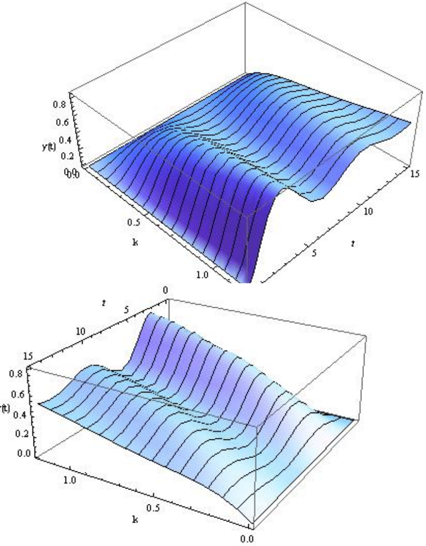


Figure: Step Response for different k

Stability

The most basic property is **Stability**:

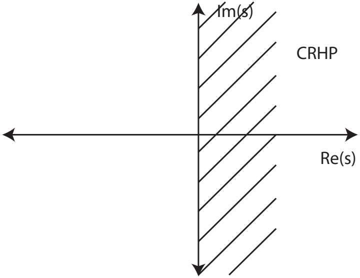
Definition 1.

A system, G is **Stable** if there exists a $K > 0$ such that

$$\|Gu\|_{L_2} \leq K\|u\|_{L_2}$$

Note: Although this is the true definition for systems defined by transfer functions, it is rarely used.

- Bounded input means bounded output.
- Stable is $y(t) \to 0$ when $u(t) \to 0$.



Definition 2.

The **Closed Right Half-Place**, $CRHP$ is the set of complex numbers with non-negative real part.

$$\{s \in \mathbb{C} : \text{Real}(s) \geq 0\}$$

Theorem 3.

A system G is stable if and only if it's transfer function $\hat{G}$ has no poles in the Closed Right Half Plane.

- Check stability by checking poles.
- x is a pole
- o is a zero

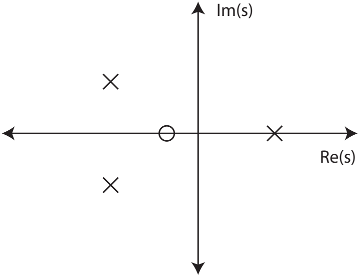


Figure: Unstable

Predicting Steady-State Error

Definition 4.

Steady-State Error for a stable system is the final difference between input and output.

$$e_{ss} = \lim_{t \to \infty} u(t) - y(t)$$

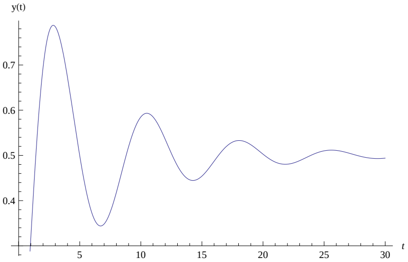
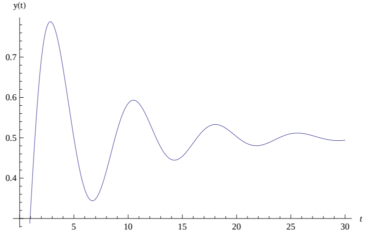


Figure: Suspension Response for $k = 1$

- Usually measured using the step response.
  - Since $u(t) = 1$,
    $e_{ss} = 1 - \lim_{t \to \infty} y(t)$

Predicting Steady-State Error



Recall: For any system G, by partial fraction expansion:

$$\hat{y}(s) = \hat{G}(s)\frac{1}{s} = \frac{r_0}{s} + \frac{r_1}{s - p_1} + \ldots + \frac{r_n}{s - p_n}$$

So

$$y(t) = r_0 \mathbf{1}(t) + r_1 e^{p_1 t} + \ldots + r_n e^{p_n t}$$

which means

$$\lim_{t \to \infty} y(t) = r_0$$

and hence

$$e_{ss} = 1 - r_0$$

Predicting Steady-State Error

The steady-state error is given by r_0.

$$e_{ss} = 1 - r_0$$

Recall: The residue at $s = 0$ is r_0 and is found as

$$r_0 = \hat{G}(s)|_{s=0} = \lim_{s \to 0} \hat{G}(s)$$

Thus the steady-state error is

$$e_{ss} = 1 - \lim_{s \to 0} \hat{G}(s)$$

This can be generalized to find the limit of any signal:

Theorem 5 (Final Value Theorem).

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\hat{y}(s)$$

- Assumes the limit exists (Stability)
- Can be used to find response to other inputs
  - Ramp, impulse, etc.

Predicting Steady-State Error

Numerical Example

$$\hat{G}(s) = \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}$$

The steady-state response is

$$\begin{aligned} y_{ss} &= \lim_{s\to 0} s\hat{y}(s) = \lim_{s\to 0} \hat{G}(s) \\ &= \lim_{s\to 0} \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)} \\ &= \frac{k}{1+k} \end{aligned}$$

The steady-state error is

$$\begin{aligned} e_{ss} &= 1 - y_{ss} = 1 - \frac{k}{1+k} \\ &= \frac{1}{1+k} \end{aligned}$$

- When $k = 0$, $e_{ss} = 1$
- As $k \to \infty$, $e_{ss} = 0$

By now, you know that motion is dominated by the poles!

- Simplify the response by considering response of each pole.
- Allows quantitative analysis

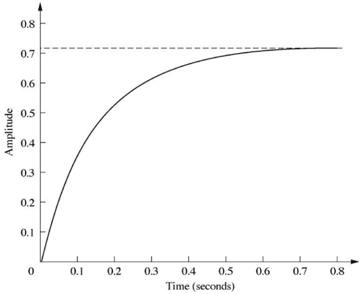


Figure: Real Pole

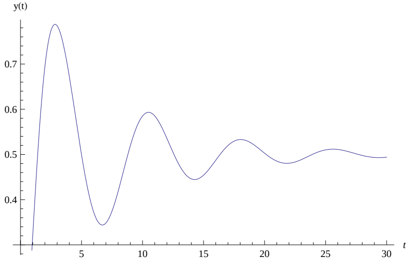


Figure: Complex Pair of Poles

We start with **Real Poles**

Consider a real pole step response:

$$\hat{y}(s) = \frac{r}{s-p}\frac{1}{s} = \frac{\frac{r}{p}}{s-p} - \frac{\frac{r}{p}}{s} \qquad y(t) = \frac{r}{p}\left(e^{pt} - 1\right)$$

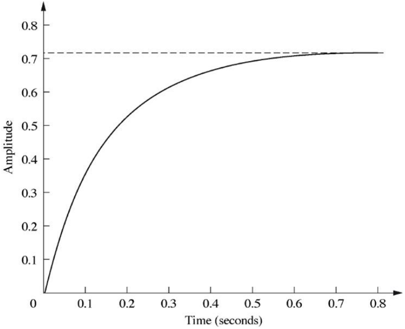
- Assume stable, so $p < 0$

Cases:

- $p > 0$ implies $y(t) \to \infty$
- $p < 0$ implies $y(t) \to -\frac{r}{p}$

Steady-State Error:

$$e_{ss} = 1 - \frac{r}{p}$$



Step Response Characteristics

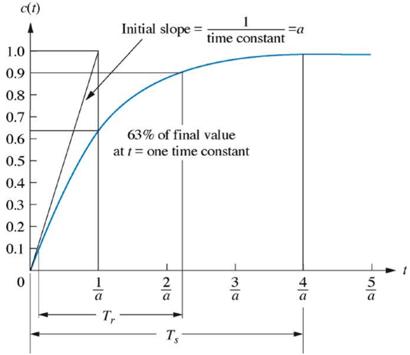
Rise Time

Besides the final value:

- How quickly will the system respond?

Definition 6.

The rise time, T_r, is the time it takes to go from .1 to .9 of the final value.



t_1 when $y(t_1) = -.1\frac{r}{p}$ is found as

$$-.1 = e^{pt_1} - 1$$

$$\ln(1 - .1) = pt_1$$

$$t_1 = \frac{\ln .9}{p} = \frac{.11}{-p}$$

Likewise for $y(t_2) = -.9\frac{r}{p}$ we get

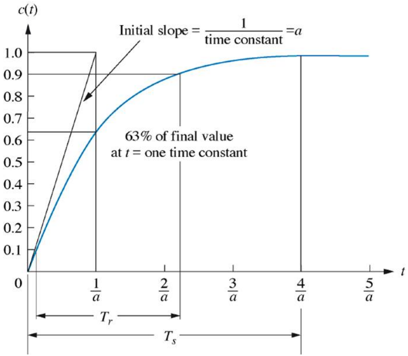
$$t_2 = \frac{\ln .1}{p} = \frac{2.31}{-p}$$

Thus rise time for a **Single Pole** is:

$$T_r = t_2 - t_1 = \frac{2.31}{-p} - \frac{.11}{-p} = \frac{2.2}{-p}$$

Will it stay there:

- How fast does it converge?
- More important for complex poles.



Definition 7.

The **Settling Time**, T_s, is the time it takes to reach and stay within .99 of the final value.

The time at $y(T_s) = -.99\frac{r}{p}$ is found as

$$-.99 = e^{pT_s} - 1$$
$$\ln(.01) = pT_s$$
$$T_s = \frac{\ln .01}{p} = -\frac{4.6}{p}$$

The settling time for a **Single Pole** is:

$$T_s = \frac{4.6}{-p}$$

Solution for Complex Poles

$$\hat{y}(s) = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + \omega_d^2 + \sigma^2}\frac{1}{s} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{k_1 s + k_2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{r_2}{s}$$

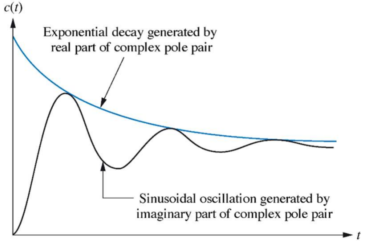
The poles are at $s = \sigma \pm \omega_d \imath$ and $s = 0$. The solution is:

$$\begin{aligned} y(t) &= 1 - e^{\sigma t}\left(\cos(\omega_d t) - \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \\ &= 1 - e^{\sigma t}\frac{\omega_n}{\omega_d}\sin\left(\omega_d t + \phi\right) \end{aligned}$$

Where $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$.

The result is oscillation with an Exponential Envelope.

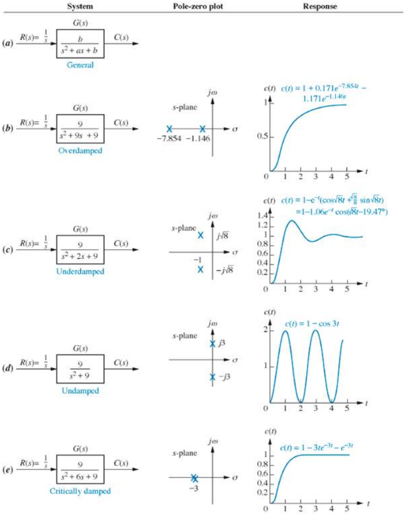
- Envelope decays at rate σ
- Speed of oscillation is ω_d, the **Damped Frequency**



Damping

We use several adjectives to describe exponential decay:

- Undamped
  - Oscillation continues forever, $\sigma = 0$
- Underdamped
  - Oscillation continues for many cycles.
- Damped
- Critically Damped
  - No oscillation or overshoot. $\omega = 0$



Besides ω, there is another way to measure oscillation

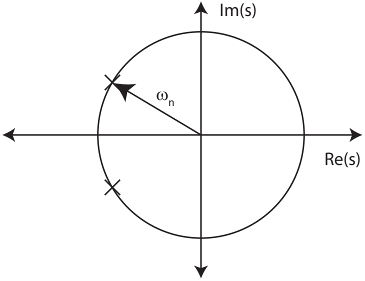
Definition 8.

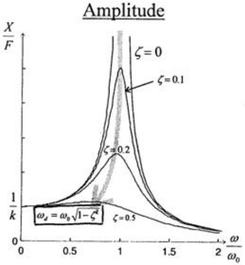
The **Natural Frequency** of a pole at $p = \sigma + \imath\omega_d$ is $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\omega_n = \sqrt{b}$.
- Radius of the pole in complex plane.

Frequency of least damping.

- Also known as resonant frequency



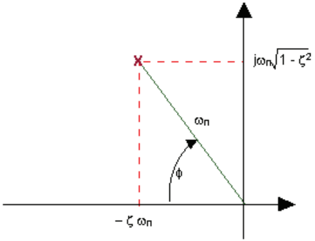


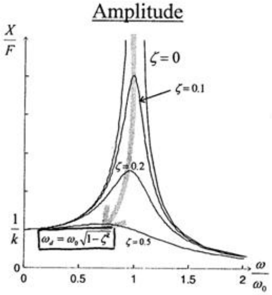
Besides σ, there are other ways to measure damping

Definition 9.

The **Damping Ratio** of a pole at $p = \sigma + \imath\omega$ is $\zeta = \frac{|\sigma|}{\omega_n}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\zeta = \frac{a}{2\omega_n}$.
- The amount the amplitude decreases per oscillation.
- The angle that the pole makes in the complex plane.





Summary

What have we learned today?

Characteristics of the Response

Real Poles

- Steady-State Error
- Rise Time
- Settling Time

Complex Poles

- Complex Pole Locations
- Damped/Natural Frequency
- Damping and Damping Ratio

Continued in Next Lecture