INFLUENCE COEFFICIENTS

16. Determine the influence coefficients of the three-degree-of-freedom spring-mass system as shown in Fig. 3-23 below.

From definition, influence coefficient α_{ij} is the deflection at the coordinate i due to a unit force applied at coordinate j. For a three-degree-of-freedom system, there will be nine influence coefficients. They are $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}$, and α_{33}.

When a unit force is applied to mass $4m$ as shown in Fig. 3-23(a), the spring of stiffness $3k$ will stretch $1/3k$, equal to α_{11}; thus $\alpha_{11} = 1/3k$.

When mass $4m$ deflects $\alpha_{11} = 1/3k$ under the action of a unit force, then masses $2m$ and m will simply move downward by the same amount, i.e.,

$$\alpha_{21} = \alpha_{31} = \alpha_{11} = 1/3k$$

By Maxwell's reciprocal principle, $\alpha_{ij} = \alpha_{ji}$. Hence $\alpha_{31} = \alpha_{13}$, $\alpha_{12} = \alpha_{21}$, and so

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{31} = 1/3k$$

To find α_{22}, apply a unit force to mass $2m$ as shown in Fig. 3-23(b). The two springs $3k$ and k are in series, and their equivalent spring constant is given by

$$1/k_{eq} = 1/3k + 1/k \quad \text{or} \quad k_{eq} = 3k/4$$

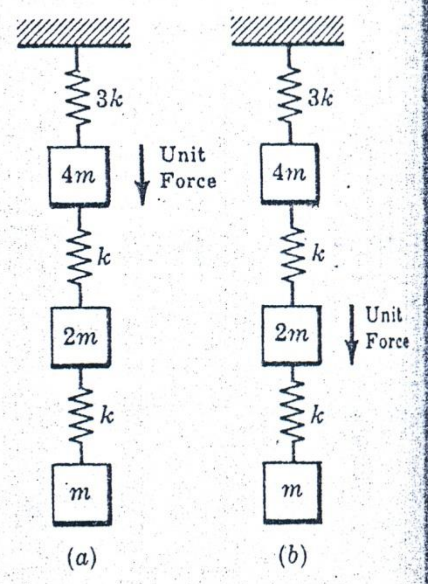

Fig. 3-23

The deflection is F/k_{eq}, or $1/(3k/4) = 4/3k = \alpha_{22}$; and as mass m hangs on mass $2m$, $\alpha_{32} = \alpha_{22}$. Now $\alpha_{32} = \alpha_{23}$ and hence

$$\alpha_{22} = \alpha_{23} = \alpha_{32}$$

To find α_{33}, apply a unit force to mass m. The three springs are in series and their equivalent spring stiffness is given by

$$1/k_{eq} = 1/3k + 1/k + 1/k = 7/3k \quad \text{or} \quad k_{eq} = 3k/7$$

and

$$\alpha_{33} = F/k_{eq} = 1/(3k/7) = 7/3k$$

The influence coefficients of the system are

$$\alpha_{11} = 1/3k, \quad \alpha_{12} = 1/3k, \quad \alpha_{13} = 1/3k$$
$$\alpha_{21} = 1/3k, \quad \alpha_{22} = 4/3k, \quad \alpha_{23} = 4/3k$$
$$\alpha_{31} = 1/3k, \quad \alpha_{32} = 4/3k, \quad \alpha_{33} = 7/3k$$

17. Determine the influence coefficients of the triple pendulum of lengths L_1, L_2, L_3 and masses m_1, m_2, m_3 as shown in Fig. 3-24.

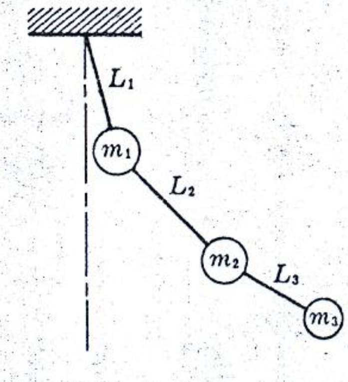

Fig. 3-24

Apply a unit horizontal force to the mass m_1 of the pendulum as shown in Fig. 3-25 below and write force equations about mass m_1. Since m_1 is in equilibrium,

$$T \sin\theta = 1 \qquad (1)$$

$$T \cos\theta = g(m_1 + m_2 + m_3) \qquad (2)$$

Divide equation (1) by (2) to obtain

$$\tan\theta = 1/g(m_1 + m_2 + m_3)$$

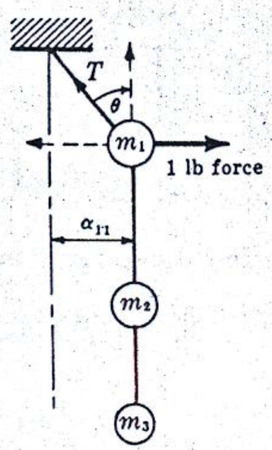

Fig. 3-25

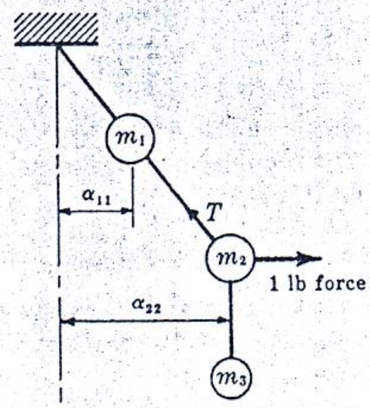

Fig. 3-26

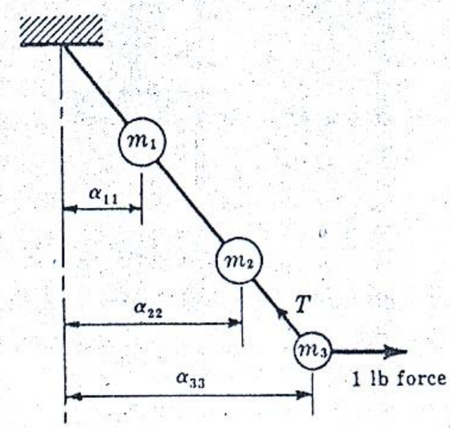

Fig. 3-27

For small angles of oscillation, $\tan\theta \doteq \sin\theta$; and from the configuration of the syste $\sin\theta = \alpha_{11}/L_1$. Hence

$$\alpha_{11} = L_1/g(m_3 + m_2 + m_1)$$

and $\alpha_{11} = \alpha_{21} = \alpha_{31}$ from the geometry of the system.

When a unit horizontal force is applied to m_2 as shown in Fig. 3-26 above, mass m_1 will displaced a distance α_{11}, but m_2 and m_3 will each be displaced an additional distance equal $L_2/g(m_2 + m_3)$. Therefore,

$$\alpha_{12} = \alpha_{11} \qquad \text{and} \qquad \alpha_{22} = \alpha_{32} = \alpha_{11} + L_2/g(m_2 + m_3)$$

Similarly, when a unit horizontal force is the only force acting on mass m_3 as shown in Fig. 3- above, mass m_1 will be simply displaced a distance α_{11}, and m_2 a distance $[\alpha_{11} + L_2/g(m_2 + m$ while m_3 will be displaced an additional distance equal to L_3/gm_3; then

$$\alpha_{13} = \alpha_{11}, \qquad \alpha_{23} = \alpha_{22}, \qquad \alpha_{33} = \alpha_{22} + L_3/gm_3$$

Thus the influence coefficients are given by

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = \frac{L_1}{g(m_1 + m_2 + m_3)}$$

$$\alpha_{21} = \frac{L_1}{g(m_1 + m_2 + m_3)}, \qquad \alpha_{22} = \alpha_{23} = \frac{L_1}{g(m_1 + m_2 + m_3)} + \frac{L_2}{g(m_2 + m_3)}$$

$$\alpha_{31} = \frac{L_1}{g(m_1 + m_2 + m_3)}, \qquad \alpha_{32} = \frac{L_1}{g(m_1 + m_2 + m_3)} + \frac{L_2}{g(m_2 + m_3)},$$

$$\alpha_{33} = \frac{L_1}{g(m_1 + m_2 + m_3)} + \frac{L_2}{g(m_2 + m_3)} + \frac{L}{gm}$$

18. Calculate the influence coefficients of the three-degree-of-freedom spring-mass system as shown in Fig. 3-28, where all the masses are equal to m and all the springs equal to k.

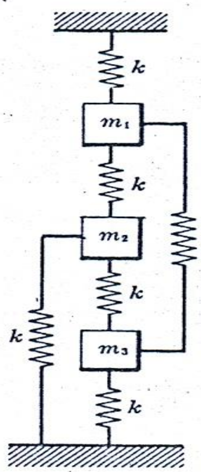

Fig. 3-28

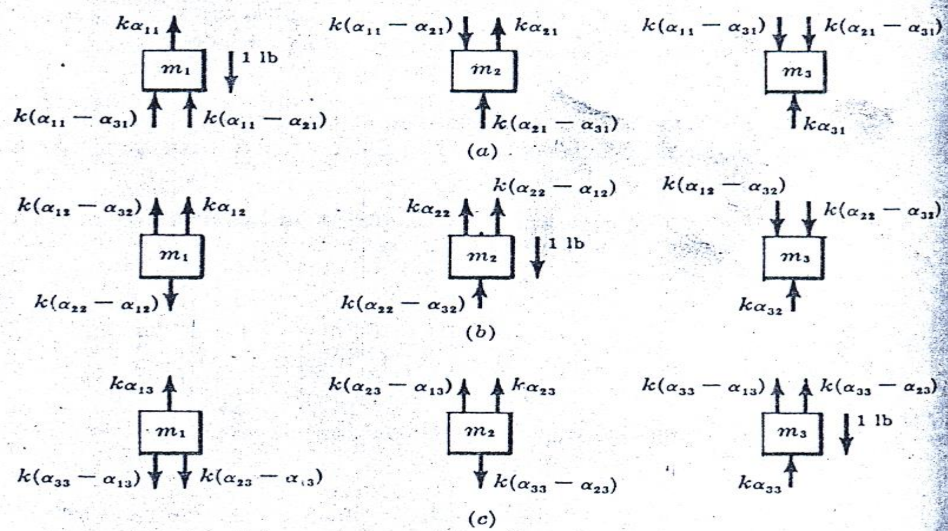

Fig. 3-29

Designate the masses as m_1, m_2, and m_3. Apply a one lb unit force to m_1. From the free-body force diagrams, Fig. 3-29(a),

$$k\alpha_{11} + k(\alpha_{11} - \alpha_{31}) + k(\alpha_{11} - \alpha_{21}) = 1 \qquad 3k\alpha_{11} - k\alpha_{21} - k\alpha_{31} = 1$$
$$k(\alpha_{11} - \alpha_{21}) = k(\alpha_{21} - \alpha_{31}) + k\alpha_{21} \qquad \text{or} \qquad 3\alpha_{21} - \alpha_{31} - \alpha_{11} = 0$$
$$k(\alpha_{21} - \alpha_{31}) + k(\alpha_{11} - \alpha_{31}) = k\alpha_{31} \qquad 3\alpha_{31} - \alpha_{11} - \alpha_{21} = 0$$

which gives

$$\alpha_{11} = 1/2k, \quad \alpha_{21} = 1/4k, \quad \alpha_{31} = 1/4k$$

Similarly, the following force equations will be obtained when a unit force is applied to mass m_2 as shown in Fig. 3-29(b) above:

$$k(\alpha_{12} - \alpha_{32}) + k\alpha_{12} = k(\alpha_{22} - \alpha_{12}) \qquad 3\alpha_{12} - \alpha_{22} - \alpha_{32} = 0$$
$$k(\alpha_{22} - \alpha_{12}) + k\alpha_{22} + k(\alpha_{22} - \alpha_{32}) = 1 \qquad \text{or} \qquad 3k\alpha_{22} - k\alpha_{32} - k\alpha_{12} = 1$$
$$k(\alpha_{22} - \alpha_{32}) + k(\alpha_{12} - \alpha_{32}) = k\alpha_{32} \qquad 3\alpha_{32} - \alpha_{12} - \alpha_{22} = 0$$

from which

$$\alpha_{12} = 1/4k, \quad \alpha_{22} = 1/2k, \quad \alpha_{32} = 1/4k$$

And when a unit force is applied to mass m_3 as shown in Fig. 3-29(c) above, we obtain

$$k(\alpha_{33} - \alpha_{13}) + k(\alpha_{23} - \alpha_{13}) = k\alpha_{13} \qquad 3\alpha_{13} - \alpha_{23} - \alpha_{33} = 0$$
$$k(\alpha_{23} - \alpha_{13}) + k\alpha_{23} = k(\alpha_{33} - \alpha_{23}) \qquad \text{or} \qquad 3\alpha_{23} - \alpha_{33} - \alpha_{13} = 0$$
$$k(\alpha_{33} - \alpha_{13}) + k(\alpha_{33} - \alpha_{23}) + k\alpha_{33} = 1 \qquad 3k\alpha_{33} - k\alpha_{13} - k\alpha_{23} = 1$$

from which

$$\alpha_{13} = 1/4k, \quad \alpha_{23} = 1/4k, \quad \alpha_{33} = 1/2k$$

The influence coefficients of the system are then

$$\alpha_{11} = 1/2k \qquad \alpha_{12} = 1/4k \qquad \alpha_{13} = 1/4k$$
$$\alpha_{21} = 1/4k \qquad \alpha_{22} = 1/2k \qquad \alpha_{23} = 1/4k$$
$$\alpha_{31} = 1/4k \qquad \alpha_{32} = 1/4k \qquad \alpha_{33} = 1/2k$$

19. Calculate the influence coefficients of a dynamic system consisting of three equal masses attached to a taut string as shown in Fig. 3-30.

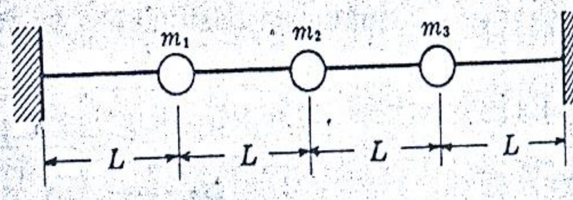

Fig. 3-30

The tension T in the string can be assumed to remain unchanged for small angles of oscillation. α_{11} is the deflection at position 1 due to a unit force applied to position 1.

At the position shown in Fig. 3-31, the unit force is balanced by the tension forces exerted by the string. For small angles of oscillation this can be written as

$$(\alpha_{11}/L)T + (\alpha_{11}/3L)T = 1$$

which gives $\alpha_{11} = 3L/4T$.

α_{21} and α_{31} are the deflections of the masses m_2 and m_3 due to a unit force applied to m_1. They are given by

$$\alpha_{21} = \tfrac{2}{3}(\alpha_{11}) = L/2T, \qquad \alpha_{31} = \tfrac{1}{3}(\alpha_{11}) = L/4T$$

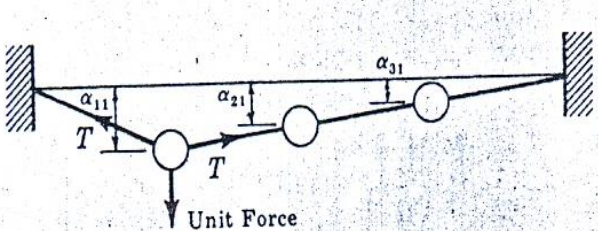

Fig. 3-31

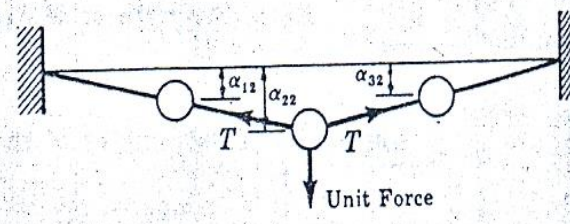

Fig. 3-32

To determine α_{22}, apply a unit force to mass m_2 as shown in Fig. 3-32. The forces acting on mass m_2 are the applied unit force and the tension forces; then

$$(\alpha_{22}/2L)T + (\alpha_{22}/2L)T = 1$$

which gives $\alpha_{22} = L/T$, and $\alpha_{12} = \alpha_{32} = L/2T$.

By symmetry, $\alpha_{11} = \alpha_{33} = 3L/4T$; and by Maxwell's reciprocal theorem, $\alpha_{12} = \alpha_{21}$, $\alpha_{13} = \alpha_{31}$, $\alpha_{23} = \alpha_{32}$. Thus the influence coefficients of the system are

$$\begin{aligned}
\alpha_{11} &= 3L/4T, & \alpha_{12} &= L/2T, & \alpha_{13} &= L/4T \\
\alpha_{21} &= L/2T, & \alpha_{22} &= L/T, & \alpha_{23} &= L/2T \\
\alpha_{31} &= L/4T, & \alpha_{32} &= L/2T, & \alpha_{33} &= 3L/4T
\end{aligned}$$

20. Prove Maxwell's reciprocal theorem $\alpha_{ij} = \alpha_{ji}$ for the simply supported beam with two concentrated loads acting as shown in Fig. 3-33.

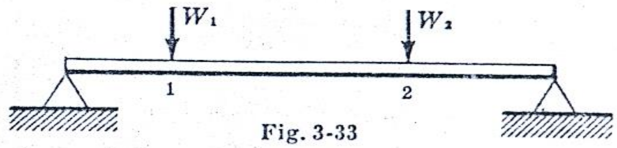

Fig. 3-33

The four influence coefficients of the system are $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$. It is necessary to show that $\alpha_{12} = \alpha_{21}$ in order to prove Maxwell's reciprocal theorem. This can be done by applying the loads in two cycles.

For the first cycle, apply W_1 first and then W_2. When W_1 is alone at position 1, the influence coefficients are α_{11}, α_{21} and

$$\text{P.E.} = \tfrac{1}{2}W_1^2\alpha_{11}$$

When W_2 is applied after W_1 is on, the additional energy of the system is $\frac{1}{2}W_2^2\alpha_{22} + W_1(W_2\alpha_{12})$ and the total energy is therefore $\frac{1}{2}W_1^2\alpha_{11} + \frac{1}{2}W_2^2\alpha_{22} + W_1(W_2\alpha_{12})$.

For the second cycle, apply W_2 first and then W_1. In a similar fashion, the total energy of the system is given by $\frac{1}{2}W_2^2\alpha_{22} + \frac{1}{2}W_1^2\alpha_{11} + W_2(W_1\alpha_{21})$.

Since at the ends of both cycles of application of loads the same state prevails, the two energy expressions must be the same. Thus by equating the two energy expressions, we obtain $\alpha_{12} = \alpha_{21}$.

It can be shown that the Maxwell's reciprocal theorem can be extended to systems with several loads acting.

21. In Fig. 3-34, assume the beam is weightless and has constant flexible rigidity EI. Use influence coefficients to determine the differential equations of motion.

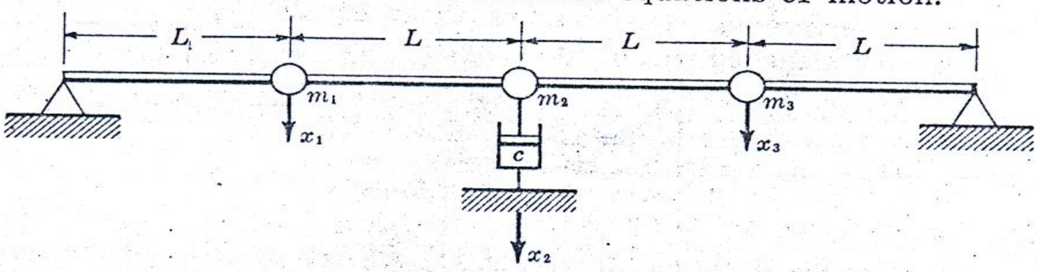

Fig. 3-34

From influence coefficient theory, the total deflections at positions 1, 2, and 3 are given by

$$x_1 = -m_1\ddot{x}_1\alpha_{11} - m_2\ddot{x}_2\alpha_{12} - m_3\ddot{x}_3\alpha_{13} - c\dot{x}_2\alpha_{12}$$
$$x_2 = -m_1\ddot{x}_1\alpha_{21} - m_2\ddot{x}_2\alpha_{22} - m_3\ddot{x}_3\alpha_{23} - c\dot{x}_2\alpha_{22}$$
$$x_3 = -m_1\ddot{x}_1\alpha_{31} - m_2\ddot{x}_2\alpha_{32} - m_3\ddot{x}_3\alpha_{33} - c\dot{x}_2\alpha_{32}$$

From Strength of Materials,

$$\alpha_{11} = \frac{9L^3}{12EI}, \qquad \alpha_{21} = \frac{11L^3}{12EI}, \qquad \alpha_{22} = \frac{16L^3}{12EI}$$

and from the symmetry of the system

$$\alpha_{33} = \alpha_{11} = \frac{9L^3}{12EI}, \qquad \alpha_{32} = \alpha_{12} = \frac{11L^3}{12EI}, \qquad \alpha_{13} = \alpha_{31} = \frac{7L^3}{12EI}$$

Fig. 3-35

Finally, by Maxwell's reciprocal theorem, $\alpha_{12} = \alpha_{21}$ and $\alpha_{23} = \alpha_{32}$. Thus the equations of motion take the following final form:

$$(9m_1\ddot{x}_1 + 11m_2\ddot{x}_2 + 11c\dot{x}_2 + 7m_3\ddot{x}_3)(L^3/12EI) + x_1 = 0$$
$$(16m_2\ddot{x}_2 + 16c\dot{x}_2 + 11m_3\ddot{x}_3 + 11m_1\ddot{x}_1)(L^3/12EI) + x_2 = 0$$
$$(9m_3\ddot{x}_3 + 7m_1\ddot{x}_1 + 11m_2\ddot{x}_2 + 11c\dot{x}_2)(L^3/12EI) + x_3 = 0$$

MATRIX ITERATION

22. Use matrix iteration to determine the natural frequencies of the system shown in Fig. 3-36.

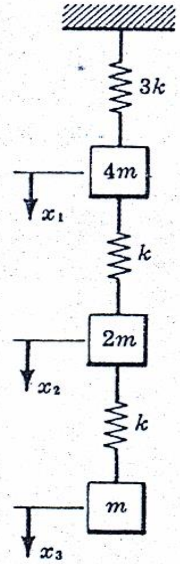

Fig. 3-36

From influence coefficient theory, the equation of motion can be written as

$$-x_1 = \alpha_{11}4m\ddot{x}_1 + \alpha_{12}2m\ddot{x}_2 + \alpha_{13}m\ddot{x}_3$$
$$-x_2 = \alpha_{21}4m\ddot{x}_1 + \alpha_{22}2m\ddot{x}_2 + \alpha_{23}m\ddot{x}_3$$
$$-x_3 = \alpha_{31}4m\ddot{x}_1 + \alpha_{32}2m\ddot{x}_2 + \alpha_{33}m\ddot{x}_3$$

When $\ddot{x}_i$ is replaced by $-\omega^2 x_i$, the equations take the form

$$x_1 = 4\alpha_{11}mx_1\omega^2 + 2\alpha_{12}mx_2\omega^2 + \alpha_{13}mx_3\omega^2$$
$$x_2 = 4\alpha_{21}mx_1\omega^2 + 2\alpha_{22}mx_2\omega^2 + \alpha_{23}mx_3\omega^2$$
$$x_3 = 4\alpha_{31}mx_1\omega^2 + 2\alpha_{32}mx_2\omega^2 + \alpha_{33}mx_3\omega^2$$

In matrix notation, this becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 m \begin{bmatrix} 4\alpha_{11} & 2\alpha_{12} & \alpha_{13} \\ 4\alpha_{21} & 2\alpha_{22} & \alpha_{23} \\ 4\alpha_{31} & 2\alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The values for the influence coefficients were found in Problem 16, Page 88, to be

$$\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{13} = \alpha_{31} = 1/3k, \qquad \alpha_{22} = \alpha_{32} = \alpha_{23} = 4/3k, \qquad \alpha_{33} = 7/3k$$

When these values are substituted into the matrix equation, we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To start the iteration process, estimate the configuration of the first mode. Let $x_1 = 1$, $x_2 = 2$, $x_3 = 4$.

First iteration:

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 12 \\ 36 \\ 48 \end{bmatrix} = \frac{\omega^2 m}{3k}(12) \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 14.0 \\ 44.0 \\ 56.0 \end{bmatrix} = \frac{\omega^2 m}{3k}(14) \begin{bmatrix} 1.0 \\ 3.2 \\ 4.0 \end{bmatrix}$$

Third iteration:

$$\begin{bmatrix} 1 \\ 3.2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3.2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 14.4 \\ 45.6 \\ 57.6 \end{bmatrix} = \frac{\omega^2 m}{3k}(14.4) \begin{bmatrix} 1.00 \\ 3.18 \\ 4.00 \end{bmatrix}$$

Since the ratio obtained here is very close to the initial value,

$$\begin{bmatrix} 1.0 \\ 3.2 \\ 4.0 \end{bmatrix} = \frac{14.4m\omega^2}{3k} \begin{bmatrix} 1.00 \\ 3.18 \\ 4.00 \end{bmatrix} \quad \text{or} \quad 1 = (14.4m\omega^2)/3k \quad \text{and} \quad \omega_1 = 0.46\sqrt{k/m} \text{ rad/sec}$$

To obtain the second principal mode, the orthogonality principle is used:

$$m_1 A_1 A_2 + m_2 B_1 B_2 + m_3 C_1 C_2 = 0$$

For the first and second mode, this becomes

$$4m(1)A_2 + 2m(3.2)B_2 + m(4)C_2 = 0$$

or
$$A_2 = -1.6B_2 - C_2, \qquad B_2 = B_2, \qquad C_2 = C_2$$

and in matrix form

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -1.6 & -1.0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}.$$

When this is combined with the matrix equation for first mode, it will converge to second moc

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -1.6 & -1.0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 0 & -4.4 & -3 \\ 0 & 1.6 & 0 \\ 0 & 1.6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Due to symmetry of the problem, the second mode is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. When this is used to start the i process, we have

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 0 & -4.4 & -3 \\ 0 & 1.6 & 0 \\ 0 & 1.6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

which repeats itself. Hence

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{3m\omega^2}{3k} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{or} \quad 1 = (\omega^2 m)/k \quad \text{and} \quad \omega_2 = \sqrt{k/m} \text{ rad/sec}$$

To obtain the third mode, write the orthogonality principle as

$$m_1 A_2 A_3 + m_2 B_2 B_3 + m_3 C_2 C_3 = 0$$
$$m_1 A_1 A_3 + m_2 B_1 B_3 + m_3 C_1 C_3 = 0$$

Substituting $A_1 = 1.0$, $B_1 = 3.2$, $C_1 = 4.0$, $A_2 = 1$, $B_2 = 0$, and $C_2 = -1$, into the orthogonality equ we obtain

$$4m(1)A_3 + 2m(0)B_3 + m(-1)C_3 = 0$$
$$4m(1)A_3 + 2m(3.20)B_3 + m(4)C_3 = 0$$

from which $A_3 = 0.25C_3$ and $B_3 = -0.78C_3$. Then

$$\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.78 \\ 0 & 0 & 1.00 \end{bmatrix} \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}$$

and when this is combined with the matrix equation for the second mode, it will yield the third

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 0 & -4.4 & -3 \\ 0 & 1.6 & 0 \\ 0 & 1.6 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -.78 \\ 0 & 0 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 0 & 0 & 0.43 \\ 0 & 0 & -1.25 \\ 0 & 0 & 1.75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k}(1.75) \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -.72 \\ 0 & 0 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assuming any arbitrary values for the third mode, it can be shown that the same third mode will be found. Further iteration is therefore not necessary. Thus

$$1 = (\omega^2 m/3k)(1.75) \quad \text{or} \quad \omega_3 = 1.32\sqrt{k/m} \text{ rad/sec}$$

. Use matrix iteration to determine the natural frequencies of the triple pendulum shown in Fig. 3-37 below.

From influence coefficient theory, the equations of motion are given by

$$-x_1 = \alpha_{11}m_1\ddot{x}_1 + \alpha_{12}m_2\ddot{x}_2 + \alpha_{13}m_3\ddot{x}_3$$
$$-x_2 = \alpha_{21}m_1\ddot{x}_1 + \alpha_{22}m_2\ddot{x}_2 + \alpha_{23}m_3\ddot{x}_3$$
$$-x_3 = \alpha_{31}m_1\ddot{x}_1 + \alpha_{32}m_2\ddot{x}_2 + \alpha_{33}m_3\ddot{x}_3$$

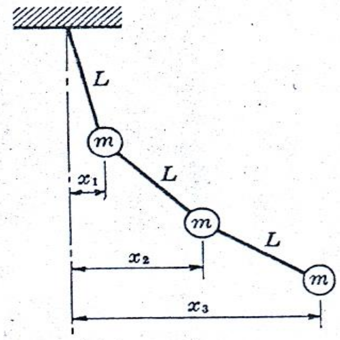

Fig. 3-37

Replacing $\ddot{x}_i$ by $-\omega^2 x_i$, the equations take the form

$$x_1 = \alpha_{11}m_1x_1\omega^2 + \alpha_{12}m_2x_2\omega^2 + \alpha_{13}m_3x_3\omega^2$$
$$x_2 = \alpha_{21}m_1x_1\omega^2 + \alpha_{22}m_2x_2\omega^2 + \alpha_{23}m_3x_3\omega^2$$
$$x_3 = \alpha_{31}m_1x_1\omega^2 + \alpha_{32}m_2x_2\omega^2 + \alpha_{33}m_3x_3\omega^2$$

To obtain numerical values for the influence coefficients, put $L_1 = L_2 = L_3 = L$ and $m_1 = m_2 = m_3 = m$ in Problem 17, Page 89:

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = L/3mg$$
$$\alpha_{21} = L/3mg, \quad \alpha_{22} = \alpha_{23} = 5L/6mg$$
$$\alpha_{31} = L/3mg, \quad \alpha_{32} = 5L/6mg, \quad \alpha_{33} = 11L/6mg$$

In matrix notation, then, the equations become

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Begin the iteration process by an arbitrary assumption for the first mode of the system.

First iteration:

$$\begin{bmatrix} 0.2 \\ 0.6 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.6 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 3.6 \\ 8.4 \\ 14.4 \end{bmatrix} = \frac{L\omega^2}{6g}(14.4)\begin{bmatrix} 0.25 \\ 0.58 \\ 1.00 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} 0.25 \\ 0.58 \\ 1.00 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 0.25 \\ 0.58 \\ 1.00 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 3.66 \\ 8.42 \\ 14.4 \end{bmatrix} = \frac{L\omega^2}{6g}(14.4)\begin{bmatrix} 0.25 \\ 0.58 \\ 1.00 \end{bmatrix}$$

Since the column repeats itself, the iteration process can be stopped. Then

$$1 = \frac{L\omega^2}{6g}(14.4) \qquad \text{or} \qquad \omega_1 = 0.65\sqrt{g/L} \text{ rad/sec}$$

To obtain the second mode, the first mode has to be suppressed during the iteration process. This is done by the use of the orthogonality principle:

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$

Substituting the first mode into the above equation, we obtain

$$m(0.25)x_1 + m(0.58)x_2 + m(1.0)x_3 = 0 \qquad \text{or} \qquad x_1 = -2.32x_2 - 4x_3, \qquad x_2 = x_2, \qquad x_3 = x_3$$

and in matrix form, this becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -2.32 & -4.0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

When this is combined with the fundamental matrix equation, it will yield a matrix equation that has no first mode present:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 0 & -2.3 & -4 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & -2.6 & -6 \\ 0 & 0.4 & -3 \\ 0 & 0.4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

From this matrix equation, use matrix iteration to determine the second mode.

First iteration:

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & -2.6 & -6 \\ 0 & 0.4 & -3 \\ 0 & 0.4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} -3.4 \\ -3.4 \\ 2.6 \end{bmatrix} = \frac{L\omega^2}{6g}(2.6)\begin{bmatrix} -1.3 \\ -1.3 \\ 1.0 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} -1.3 \\ -1.3 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & -2.6 & -6 \\ 0 & 0.4 & -3 \\ 0 & 0.4 & 3 \end{bmatrix}\begin{bmatrix} -1.3 \\ -1.3 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} -2.6 \\ -3.5 \\ 2.5 \end{bmatrix} = \frac{L\omega^2}{6g}(2.5)\begin{bmatrix} -1.05 \\ -1.40 \\ 1.00 \end{bmatrix}$$

Third iteration:

$$\begin{bmatrix} -1.0 \\ -1.4 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & -2.6 & -6 \\ 0 & 0.4 & -3 \\ 0 & 0.4 & 3 \end{bmatrix}\begin{bmatrix} -1.0 \\ -1.4 \\ 1.0 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} -2.4 \\ -3.5 \\ 2.5 \end{bmatrix} = \frac{L\omega^2}{6g}(2.5)\begin{bmatrix} -1.0 \\ -1.4 \\ 1.0 \end{bmatrix}$$

Since the assumed mode in the last iteration repeats itself, the iteration process can be stoppe mode of vibration and the natural frequency are therefore given by

$$\begin{bmatrix} -1.0 \\ -1.4 \\ 1.0 \end{bmatrix} \quad \text{and} \quad 1 = \frac{L\omega^2}{6g}(2.5) \quad \text{or} \quad \omega_2 = 1.52\sqrt{g/L} \text{ rad/sec}$$

In order to obtain the third principal mode and thus the third natural frequency of the both the first and second modes must not appear in the iteration process. This is again done b the orthogonality principle expressed as

$$m_1A_1A_3 + m_2B_1B_3 + m_3C_1C_3 = 0, \qquad m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$

For first and third modes, this becomes

$$m(0.25)x_1 + m(0.6)x_2 + m(1.0)x_3 = 0$$

and for second and third modes, we have

$$m(-1.0)x_1 + m(-1.4)x_2 + m(1.0)x_3 = 0 \qquad \text{or} \qquad x_1 = 8x_3, \qquad x_2 = -5x_3, \qquad x_3 =$$

and in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

When this is combined with the matrix equation for second mode, we obtain the matrix equat third mode:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & -2.6 & -6 \\ 0 & 0.4 & -3 \\ 0 & 0.4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{L\omega^2}{6g}\begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assume any convenient value for the third mode, and start the iteration process. It will b that the same mode shape $\begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$ is obtained repeatedly. This means that $\begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$ is actually tl mode of the system. Thus

$$1 = L\omega^2/6g \qquad \text{and} \qquad \omega_3 = 2.45\sqrt{g/L} \text{ rad/sec}$$

The three natural frequencies of the triple pendulum are therefore given by

$$\omega_1 = 0.65\sqrt{g/L}, \qquad \omega_2 = 1.52\sqrt{g/L}, \qquad \omega_3 = 2.45\sqrt{g/L} \text{ rad/sec}$$

I. Determine the highest natural frequency of the three-degree-of-freedom sprin system as shown in Fig. 3-38 below. Use the inverse matrix method.

As discussed earlier, the deflection equations of the masses are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{2k}\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1. Determine the highest natural frequency of the three-degree-of-freedom sprin system as shown in Fig. 3-38 below. Use the inverse matrix method.

As discussed earlier, the deflection equations of the masses are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{2k} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using inverse matrix theory, this can be written as

$$\frac{2k}{\omega^2 m} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (1$$

where $[D]^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^{-1}$ is the inverse of matrix $[D] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$.

From matrix theory, Adjoint $[D]$ can be found in the following manner:

$$\text{Adjoint } [D] = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

and

$$|D| = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}$$

Hence

$$[D]^{-1} = \frac{\text{Adjoint } [D]}{|D|} = \frac{1}{2}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

The inverse matrix $[D]^{-1}$ can also be found by the elementary operations as follow:

Operation	$[D]$			$[D]^{-1}$		
Multiply $[D]$ by a factor of 2	1 1/2 1/2	1/2 1 1/2	1/2 1/2 1	1 0 0	0 1 0	0 0 1
Row (1) minus row (2)	2 1 1	1 2 1	1 1 2	2 0 0	0 2 0	0 0 2
Row (3) minus row (2)	1 1 1	−1 2 1	0 1 2	2 0 0	−2 2 0	0 0 2
Row (2) minus row (3)	1 1 0	−1 2 −1	0 1 1	2 0 0	−2 2 −2	0 0 2
Row (2) minus row (1)	1 1 0	−1 3 −1	0 0 1	2 0 0	−2 4 −2	0 −2 2
Multiply row (2) by a factor of 1/4	1 0 0	−1 4 −1	0 0 1	2 −2 0	−2 6 −2	0 −2 2
Add row (2) to row (1)	1 0 0	−1 1 −1	0 0 1	2 −1/2 0	−2 3/2 −2	0 −1/2 2
Add row (2) to row (3)	1 0 0	0 1 −1	0 0 1	3/2 −1/2 0	−1/2 3/2 −2	−1/2 −1/2 2
	1 0 0	0 1 0	0 0 1	3/2 −1/2 −1/2	−1/2 3/2 −1/2	−1/2 −1/2 3/2

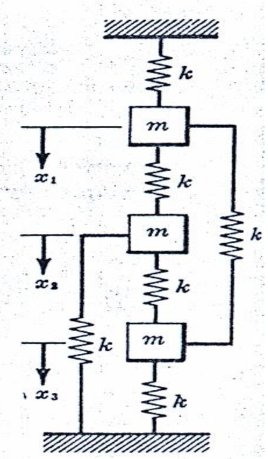

Fig. 3-38

which also gives $[D]^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

Substituting $[D]^{-1}$ into equation (*1*), we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2k}{\omega^2 m}\left(\frac{1}{2}\right)\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Assume the third mode to be $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and substitute this into equation (*2*) to obtain

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{k}{\omega^2 m}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{4k}{\omega^2 m}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The assumed mode $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ repeats itself. This means the assumed value is the third mode. Hence

$$1 = 4k/\omega^2 m \quad \text{and} \quad \omega_3 = 2\sqrt{k/m} \text{ rad/sec}$$

25. Use the Inverse Matrix method to determine the highest natural frequency of th spring-mass system as shown in Fig. 3-39.

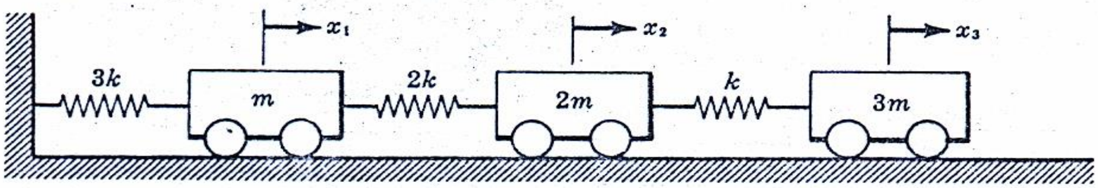

Fig. 3-39

From influence coefficient theory the equations of motion can be expressed as

$$\begin{aligned} -x_1 &= \alpha_{11} m \ddot{x}_1 + \alpha_{12} 2m \ddot{x}_2 + \alpha_{13} 3m \ddot{x}_3 \\ -x_2 &= \alpha_{21} m \ddot{x}_1 + \alpha_{22} 2m \ddot{x}_2 + \alpha_{23} 3m \ddot{x}_3 \\ -x_3 &= \alpha_{31} m \ddot{x}_1 + \alpha_{32} 2m \ddot{x}_2 + \alpha_{33} 3m \ddot{x}_3 \end{aligned}$$

where $\alpha_{11} = \alpha_{12} = \alpha_{13} = 1/3k$; $\alpha_{21} = 1/3k$, $\alpha_{22} = \alpha_{23} = 5/6k$; $\alpha_{31} = 1/3k$, $\alpha_{32} = 5/6k$, $\alpha_{33} = 11/$
Replacing $\ddot{x}_i$ by $-\omega^2 x_i$, we have

$$\begin{aligned} x_1 &= \alpha_{11} m x_1 \omega^2 + 2\alpha_{12} m x_2 \omega^2 + 3\alpha_{13} m x_3 \omega^2 \\ x_2 &= \alpha_{21} m x_1 \omega^2 + 2\alpha_{22} m x_2 \omega^2 + 3\alpha_{23} m x_3 \omega^2 \\ x_3 &= \alpha_{31} m x_1 \omega^2 + 2\alpha_{32} m x_2 \omega^2 + 3\alpha_{33} m x_3 \omega^2 \end{aligned}$$

or in matrix notation,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{6k}\begin{bmatrix} 2 & 4 & 6 \\ 2 & 10 & 15 \\ 2 & 10 & 33 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using inverse matrix theory, equation (*2*) can be written as

$$\frac{6k}{\omega^2 m}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \\ 2 & 10 & 15 \\ 2 & 10 & 33 \end{bmatrix}^{-1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $[D]^{-1} = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 10 & 15 \\ 2 & 10 & 33 \end{bmatrix}^{-1}$ is the inverse of $[D] = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 10 & 15 \\ 2 & 10 & 33 \end{bmatrix}$

Substituting $[D]^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$ into (3),

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (4)$$

Assume the third mode to be $\begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix}$ and begin the iteration process with equation (4).

First iteration:

$$\begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 58 \\ -16.5 \\ 1.7 \end{bmatrix} = \frac{k}{m\omega^2}(1.7)\begin{bmatrix} 34 \\ -9.7 \\ 1.0 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} 30 \\ -10 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 30 \\ -10 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 170 \\ -45.5 \\ 3.7 \end{bmatrix} = \frac{k}{m\omega^2}(3.7)\begin{bmatrix} 46 \\ -12.3 \\ 1 \end{bmatrix}$$

Third iteration:

$$\begin{bmatrix} 45 \\ -11 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 45 \\ -11 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 247 \\ -62 \\ 4 \end{bmatrix} = \frac{k}{m\omega^2}(4)\begin{bmatrix} 61 \\ -15 \\ 1 \end{bmatrix}$$

Fourth iteration:

$$\begin{bmatrix} 60 \\ -15 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 5 & -2 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 60 \\ -15 \\ 1 \end{bmatrix} = \frac{k}{m\omega^2}\begin{bmatrix} 330 \\ -83 \\ 5.4 \end{bmatrix} = \frac{k}{m\omega^2}(5.4)\begin{bmatrix} 60.7 \\ -15.3 \\ 1 \end{bmatrix}$$

The assumed column approximately repeats itself; this means the assumed value is correct. Hence

$$1 = \frac{k}{m\omega^2}(5.4) \qquad \text{and} \qquad \omega_3 = 2.36\sqrt{k/m} \text{ rad/sec}$$

THE STODOLA METHOD

26. Use the Stodola method to find the fundamental mode of vibration and its natural frequency of the spring-mass system as shown in Fig. 3-40. $k_1 = k_2 = k_3 = 1$ lb/in, $m_1 = m_2 = m_3 = 1$ lb-sec²/in.

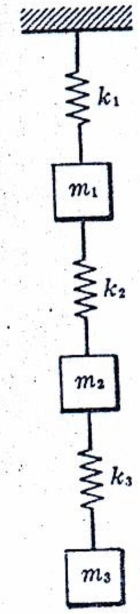

Fig. 3-40

Assume that the system is vibrating at one of its principal modes with natural frequency ω and that the motion is periodic. Then the system is acted upon by inertia forces $-m_i \ddot{x}_i$. Now

$$x_i = A_i \sin \omega t \quad \text{and} \quad -m_i \ddot{x}_i = \omega^2 m_i A_i$$

The Stodola method may be set up in the following tabular form as follows: Assuming an arbitrary set of values for the fundamental principal mode, the inertia force acting on each mass is equal to the product of the assumed deflection and the square of the natural frequency as shown in row 2. The spring force in row 3 is equal to the total inertia force acting on each spring. Row 4 is obtained by dividing row 3, term by term, by their respective spring constants. The calculated deflections in row 5 are found by adding the deflections due to the springs, with the mass near the fixed end having the least deflection and so on. The calculated deflections are then compared with the assumed deflections. This process is continued until the calculated deflections are proportional to the assumed deflections. When this is true the assumed deflections will represent the configuration of the fundamental principal mode of vibration of the system.

	k_1	m_1	k_2	m_2	k_3	m_3
1. Assumed deflection		1		1		1
2. Inertia force		ω^2		ω^2		ω^2
3. Spring force	$3\omega^2$		$2\omega^2$		ω^2	
4. Spring deflection	$3\omega^2$		$2\omega^2$		ω^2	
5. Calculated deflection		$3\omega^2$		$5\omega^2$		$6\omega^2$
		1		1.67		2
1. Assumed deflection		1		1.67		2
2. Inertia force		ω^2		$1.67\omega^2$		$2\omega^2$
3. Spring force	$4.67\omega^2$		$3.67\omega^2$		$2\omega^2$	
4. Spring deflection	$4.67\omega^2$		$3.67\omega^2$		$2\omega^2$	
5. Calculated deflection		$4.67\omega^2$		$8.34\omega^2$		$10.34\omega^2$
		1		1.79		2.21
1. Assumed deflection		1		1.79		2.21
2. Inertia force		ω^2		$1.79\omega^2$		$2.21\omega^2$
3. Spring force	$5\omega^2$		$4\omega^2$		$2.21\omega^2$	
4. Spring deflection	$5\omega^2$		$4\omega^2$		$2.21\omega^2$	
5. Calculated deflection		$5\omega^2$		$9\omega^2$		$11.21\omega^2$
		1		1.8		2.24

The assumed deflection $\begin{bmatrix} 1.00 \\ 1.79 \\ 2.21 \end{bmatrix}$ at this point is very close to the calculated deflection. Hence the fundamental principal mode of vibration is given by

$$\begin{bmatrix} 1.00 \\ 1.80 \\ 2.24 \end{bmatrix}$$

and the fundamental natural frequency is found from

$$1.00 + 1.80 + 2.24 = (5 + 9 + 11.21)\omega^2 \qquad \text{or} \qquad \omega_1 = 0.44 \text{ rad/sec}$$

27. Use the Stodola method to determine the lowest natural frequency of the four-degree-of-freedom spring-mass system as shown in Fig. 3-41.

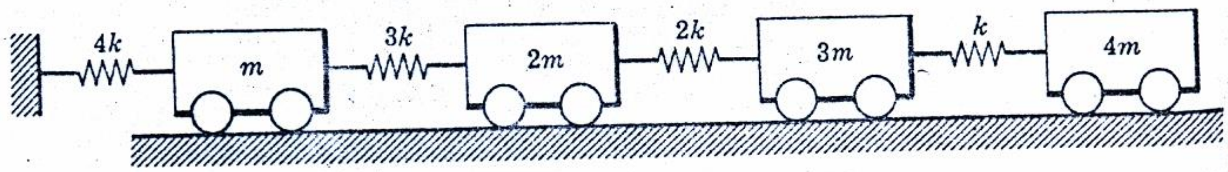

Fig. 3-41

See Problem 26 for explanation and procedure.

	$k_1 = 4k$	$m_1 = m$	$k_2 = 3k$	$m_2 = 2m$	$k_3 = 2k$	$m_3 = 3m$	$k_4 = k$	$m_4 = 4m$
Assumed deflection		4.00		3.00		2.00		1.00
Inertia force		$4\omega^2$		$6\omega^2$		$6\omega^2$		$4\omega^2$
Spring force	$20\omega^2$		$16\omega^2$		$10\omega^2$		$4\omega^2$	
Spring deflection	$5\omega^2$		$5.3\omega^2$		$5\omega^2$		$4\omega^2$	
Calculated deflection		$5\omega^2$		$10.3\omega^2$		$15.3\omega^2$		$19.3\omega^2$
Assumed deflection		1.00		2.00		3.00		4.00
Inertia force		ω^2		$4\omega^2$		$9\omega^2$		$16\omega^2$
Spring force	$30\omega^2$		$29\omega^2$		$25\omega^2$		$16\omega^2$	
Spring deflection	$7.5\omega^2$		$9.7\omega^2$		$12.5\omega^2$		$16\omega^2$	
Calculated deflection		$7.5\omega^2$		$17.2\omega^2$		$29.7\omega^2$		$45.7\omega^2$
Assumed deflection		1.00		2.00		4.00		6.00
Inertia force		ω^2		$4\omega^2$		$12\omega^2$		$24\omega^2$
Spring force	$41\omega^2$		$40\omega^2$		$36\omega^2$		$24\omega^2$	
Spring deflection	$10.25\omega^2$		$13.3\omega^2$		$18\omega^2$		$24\omega^2$	
Calculated deflection		$10.25\omega^2$		$23.55\omega^2$		$41.55\omega^2$		$65.55\omega^2$
Assumed deflection		1.00		2.2		4.00		6.4
Inertia force		ω^2		$4.4\omega^2$		$12\omega^2$		$25.6\omega^2$
Spring force	$43\omega^2$		$42\omega^2$		$37.6\omega^2$		$25.6\omega^2$	
Spring deflection	$10.75\omega^2$		$14\omega^2$		$18.8\omega^2$		$25.6\omega^2$	
Calculated deflection		$10.75\omega^2$		$24.75\omega^2$		$43.55\omega^2$		$69.15\omega^2$
		1.00		2.30		4.05		6.42

Therefore the first principal mode is given by $\begin{bmatrix} 1.00 \\ 2.30 \\ 4.05 \\ 6.42 \end{bmatrix}$ and the lowest natural frequency is obtained from

$$(1 + 2.3 + 4.05 + 6.42) = (10.75 + 24.75 + 43.55 + 69.15)\omega^2 \quad \text{or} \quad 13.77 = 148.2\omega^2$$

Hence $\omega_1^2 = 0.093$ and $\omega_1 = 0.306\sqrt{k/m}$ rad/sec

28. Prove that the Stodola method will converge to the fundamental mode of vibration.

The Stodola method begins with assumed deflections for the fundamental mode of a system. The corresponding inertia forces due to these assumed deflections are calculated. Compared with actual inertia forces and deflections of the system, the inertia forces just found will produce a new set of deflections which is used to start the next iteration. The process is repeated. Eventually, this process will converge to the fundamental mode; the degree of accuracy depends on the number of iterations.

The general motion of an n-degree-of-freedom system is given by

$$\begin{aligned} x_1 &= A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2) + \cdots + A_n \sin(\omega_n t + \psi_n) \\ x_2 &= B_1 \sin(\omega_1 t + \psi_1) + B_2 \sin(\omega_2 t + \psi_2) + \cdots + B_n \sin(\omega_n t + \psi_n) \\ x_3 &= C_1 \sin(\omega_1 t + \psi_1) + C_2 \sin(\omega_2 t + \psi_2) + \cdots + C_n \sin(\omega_n t + \psi_n) \\ &\cdots\cdots\cdots \end{aligned} \tag{1}$$

Let the assumed deflections be an arbitrary superposition of all the modes of the system, with constants $a_1, a_2, \ldots, a_n$,

$$\begin{aligned} x_1 &= a_1 A_1 + a_2 A_2 + \cdots + a_n A_n \\ x_2 &= a_1 B_1 + a_2 B_2 + \cdots + a_n B_n \\ x_3 &= a_1 C_1 + a_2 C_2 + \cdots + a_n C_n \\ &\cdots\cdots\cdots \end{aligned} \tag{2}$$

The corresponding inertia forces are

$$\begin{array}{l} m_1(a_1A_1 + a_2A_2 + \cdots + a_nA_n)\omega^2 \\ m_2(a_1B_1 + a_2B_2 + \cdots + a_nB_n)\omega^2 \\ m_3(a_1C_1 + a_2C_2 + \cdots + a_nC_n)\omega^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (3)$$

where $m_1, m_2, \ldots, m_n$ are the masses of the system and ω is the natural frequency.

Now if the system is vibrating with all the principal modes present, the inertia forces and the corresponding deflections are

$$\begin{array}{ll} m_1(A_1\omega_1^2 + A_2\omega_2^2 + \cdots + A_n\omega_n^2), & (A_1 + A_2 + \cdots + A_n) \\ m_2(B_1\omega_1^2 + B_2\omega_2^2 + \cdots + B_n\omega_n^2), & (B_1 + B_2 + \cdots + B_n) \\ m_3(C_1\omega_1^2 + C_2\omega_2^2 + \cdots + C_n\omega_n^2), & (C_1 + C_2 + \cdots + C_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \end{array} \quad (4)$$

Hence the inertia forces in (3) will produce a new set of deflections:

$$\begin{array}{l} \omega^2(a_1A_1/\omega_1^2 + a_2A_2/\omega_2^2 + \cdots + a_nA_n/\omega_n^2) \\ \omega^2(a_1B_1/\omega_1^2 + a_2B_2/\omega_2^2 + \cdots + a_nB_n/\omega_n^2) \\ \omega^2(a_1C_1/\omega_1^2 + a_2C_2/\omega_2^2 + \cdots + a_nC_n/\omega_n^2) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (5)$$

Now

$$\begin{array}{l} x_1 = \omega^2(a_1A_1/\omega_1^2 + a_2A_2/\omega_2^2 + \cdots + a_nA_n/\omega_n^2) \\ x_2 = \omega^2(a_1B_1/\omega_1^2 + a_2B_2/\omega_2^2 + \cdots + a_nB_n/\omega_n^2) \\ x_3 = \omega^2(a_1C_1/\omega_1^2 + a_2C_2/\omega_2^2 + \cdots + a_nC_n/\omega_n^2) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (6)$$

Using the deflections in (6) as the assumed deflections, and carrying out exactly the steps in the last iteration, we have

$$\begin{array}{l} x_1 = \omega^4(a_1A_1/\omega_1^4 + a_2A_2/\omega_2^4 + \cdots + a_nA_n/\omega_n^4) \\ x_2 = \omega^4(a_1B_1/\omega_1^4 + a_2B_2/\omega_2^4 + \cdots + a_nB_n/\omega_n^4) \\ x_3 = \omega^4(a_1C_1/\omega_1^4 + a_2C_2/\omega_2^4 + \cdots + a_nC_n/\omega_n^4) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (7)$$

After r iterations, the assumed deflections take the following general form:

$$\begin{array}{l} x_1 = \omega^{2r}(a_1A_1/\omega_1^{2r} + a_2A_2/\omega_2^{2r} + \cdots + a_nA_n/\omega_n^{2r}) \\ x_2 = \omega^{2r}(a_1B_1/\omega_1^{2r} + a_2B_2/\omega_2^{2r} + \cdots + a_nB_n/\omega_n^{2r}) \\ x_3 = \omega^{2r}(a_1C_1/\omega_1^{2r} + a_2C_2/\omega_2^{2r} + \cdots + a_nC_n/\omega_n^{2r}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (8)$$

or

$$\begin{array}{l} x_1 = (a_1\omega^{2r}/\omega_1^{2r})(A_1 + a_2A_2\omega_1^{2r}/a_1\omega_2^{2r} + \cdots + a_nA_n\omega_1^{2r}/a_1\omega_n^{2r}) \\ x_2 = (a_1\omega^{2r}/\omega_1^{2r})(B_1 + a_2B_2\omega_1^{2r}/a_1\omega_2^{2r} + \cdots + a_nB_n\omega_1^{2r}/a_1\omega_n^{2r}) \\ x_3 = (a_1\omega^{2r}/\omega_1^{2r})(C_1 + a_2C_2\omega_1^{2r}/a_1\omega_2^{2r} + \cdots + a_nC_n\omega_1^{2r}/a_1\omega_n^{2r}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \quad (9)$$

As $\omega_1 < \omega_2 < \omega_3 < \cdots < \omega_n$, and as the number of iterations is sufficiently large or the value of r assumes a sufficient large number, the ratios of the natural frequencies become very small. In most cases, less than ten iterations are required to obtain the fundamental mode of the system. Thus for sufficiently large numbers of iterations, the calculated deflections in (9) become

$$\begin{array}{l} x_1 = a_1A_1\omega^{2r}/\omega_1^{2r} \\ x_2 = a_1B_1\omega^{2r}/\omega_1^{2r} \\ x_3 = a_1C_1\omega^{2r}/\omega_1^{2r} \\ \cdots\cdots\cdots \end{array} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \end{bmatrix} = a_1\omega^{2r}/\omega_1^{2r} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ \cdot \end{bmatrix}$$

which approaches very closely the pure fundamental mode of vibration of the system.

Thus the Stodola method converges to the fundamental mode of vibration for an n-degree-of freedom system.

THE HOLZER METHOD

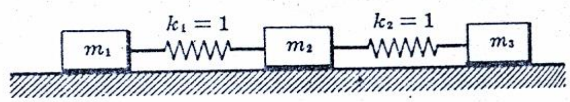

Fig. 3-42

29. Use the Holzer method to determine the natural frequencies of the spring-mass system as shown in Fig. 3-42. Here $m_1 = m_2 = m_3 = 1$ lb-sec²/in.

Begin the Holzer tabulation with the column of position, indicating the masses of the system. The second column is for the values of the different masses of the system; this information is given. The third column is the product of mass and frequency squared. Displacement comes next, and is obtained from the preceding row minus the total displacement at the end of the same row. Column five is just the product of columns three and four. The total inertia force is inserted in column six. It is equal to the sum of the total inertia force in the preceding row plus the inertia force on the same row. The rest are plainly evident.

An initial displacement, usually equal to unity for convenience, is assumed. If the assumed frequency happens to be one of the natural frequencies of the systems, the final total inertia force on the system should be zero. This is because the system is having free vibration. If the final total inertia force is not equal to zero, the amount indicates the discrepancy of the assumed frequency.

Table

Position	m_i	$m_i\omega^2$	x_i	$m_i x_i \omega^2$	$\sum_1^i m_i x_i \omega^2$	k_{ij}	$\sum_1^i m_i x_i \omega^2 / k_{ij}$
Assumed frequency, $\omega = 0.5$							
1	1	0.25	1	0.25	0.25	1	0.25
2	1	0.25	0.75	0.19	0.44	1	0.44
3	1	0.25	0.31	0.07	0.51		
Assumed frequency, $\omega = 0.75$							
1	1	0.56	1	0.56	0.56	1	0.56
2	1	0.56	0.44	0.24	0.80	1	0.80
3	1	0.56	−0.36	−0.2	0.60		
Assumed frequency, $\omega = 1.0$							
1	1	1	1	1	1	1	1
2	1	1	0	0	1	1	1
3	1	1	−1	−1	0		
Assumed frequency, $\omega = 1.5$							
1	1	2.25	1.0	2.25	2.25	1	2.25
2	1	2.25	−1.25	−2.82	−0.57	1	−0.57
3	1	2.25	−0.68	−1.53	−2.10		
Assumed frequency, $\omega = 1.79$							
1	1	3.21	1	3.21	3.21	1	3.21
2	1	3.21	−2.21	−7.08	−3.87	1	−3.87
3	1	3.21	1.66	5.34	1.47		
Assumed frequency, $\omega = 2.0$							
1	1	4	1	4	4	1	4
2	1	4	−3	−12	−8	1	−8
3	1	4	5	20	12		

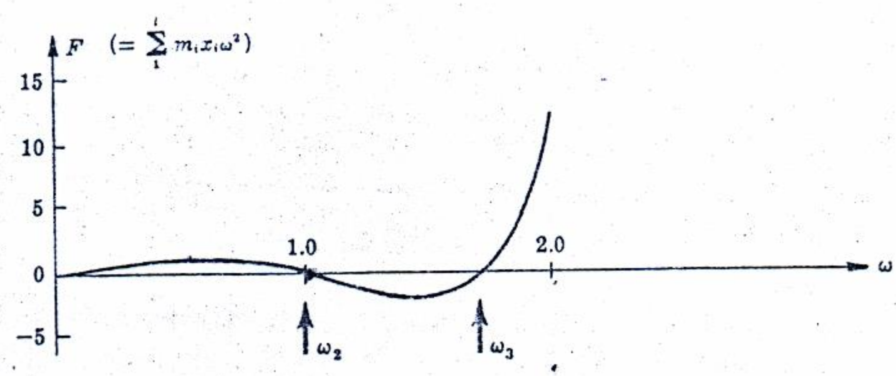

Fig. 3-43

Therefore the natural frequencies are $\omega_1 = 0$, $\omega_2 = 1.0$, $\omega_3 = 1.7$ rad/sec.

30. Use the Holzer method to determine the natural frequencies of the four-mass system as shown in Fig. 3-44, if $k = 1$ lb/in and $m = 1$ lb-sec²/in.

See procedure given in Problem 29.

Table

Item	m	$m\omega^2$	x	$mx\omega^2$	$\Sigma mx\omega^2$	k	$\Sigma mx\omega^2/k$
Assumed frequency, $\omega = 0.2$							
1	4	.16	1	.16	.16	1	.16
2	3	.12	.84	.101	.261	2	.13
3	2	.08	.71	.056	.317	3	.105
4	1	.04	.605	.025	.342	4	.0855
5	∞	∞	.5195				
Assumed frequency, $\omega = 0.3$							
1	4	.36	1	.36	.36	1	.36
2	3	.27	.64	.173	.533	2	.267
3	2	.18	.373	.067	.600	3	.200
4	1	.09	.173	.0155	.6155	4	.1539
5	∞	∞	.0192				
Assumed frequency, $\omega = 0.4$							
1	4	.64	1	.64	.64	1	.64
2	3	.48	.36	.173	.813	2	.406
3	2	.32	−.046	−.0147	.798	3	.266
4	1	.16	−.312	−.049	.748	4	.187
5	∞	∞	−.499				
Assumed frequency, $\omega = 0.6$							
1	4	1.44	1	1.44	1.44	1	1.44
2	3	1.08	−.44	−.475	.965	2	.482
3	2	.72	−.922	−.664	.301	3	.100
4	1	.36	−1.023	−.368	−.067	4	−.017
5	∞	∞	−1.006				
Assumed frequency, $\omega = 0.8$							
1	4	2.56	1	2.56	2.56	1	2.56
2	3	1.92	−1.56	−3.00	−.44	2	−.22
3	2	1.28	−1.34	−1.72	−2.16	3	−.73
4	1	.64	−.61	−.39	−2.55	4	−.64
5	∞	∞	.03				

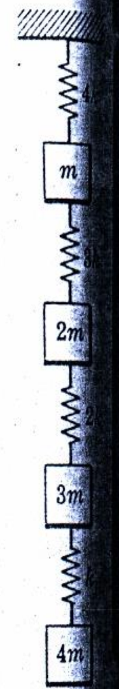

Fig. 3-4

Table (cont.)

Item	m	$m\omega^2$	x	$mx\omega^2$	$\Sigma mx\omega^2$	k	$\Sigma mx\omega^2/k$
Assumed frequency, $\omega = 1.0$							
1	4	4	1	4	4	1	4
2	3	3	−3	−9	−5	2	−2.5
3	2	2	−.5	−1	−6	3	−2.0
4	1	1	1.5	1.5	−4.5	4	−1.13
5	∞	∞	2.63				
Assumed frequency, $\omega = 1.5$							
1	4	9	1	9	9	1	9
2	3	6.75	−8	−54	−45	2	−22.5
3	2	4.5	14.5	65.3	20.3	3	6.77
4	1	2.25	7.73	17.4	37.7	4	9.43
5	∞	∞	−1.70				
Assumed frequency, $\omega = 1.8$							
1	4	12.96	1	12.96	12.96	1	12.96
2	3	9.72	−11.96	−116.4	−103.44	2	−51.72
3	2	6.48	39.76	257.7	154.26	3	51.42
4	1	3.24	−11.66	−37.8	116.46	4	29.12
5	∞	∞	−40.78				
Assumed frequency, $\omega = 2.0$							
1	4	16	1	16	16	1	16
2	3	12	−15	−180	−164	2	−82
3	2	8	67	536	372	3	124
4	1	4	−57	−228	144	4	36
5	∞	∞	−93				
Assumed frequency, $\omega = 2.5$							
1	4	25	1	25	25	1	25
2	3	18.75	−24	−450	−425	2	−212.5
3	2	12.5	188.5	2360	1935	3	645
4	1	6.25	−456.5	−2860	−925	4	−231
5	∞	∞	−225.5				
Assumed frequency, $\omega = 3.0$							
1	4	36	1	36	36	1	36
2	3	27	−35	−945	−909	2	−455
3	2	18	420	7560	6651	3	2220
4	1	9	−1800	−16,200	−9550	4	−2388
5	∞	∞	588				

Plot the curve with the assumed frequencies against the amplitudes of the fixed end as shown in Fig. 3-45. The natural frequencies of the system are given by the intersections of the curve with the frequency axis. The natural frequencies are

$$\omega_1 = 0.30 \text{ rad/sec}$$
$$\omega_2 = 0.81 \text{ rad/sec}$$
$$\omega_3 = 1.45 \text{ rad/sec}$$
$$\omega_4 = 2.83 \text{ rad/sec}$$

Note: Curve is not drawn to scale.

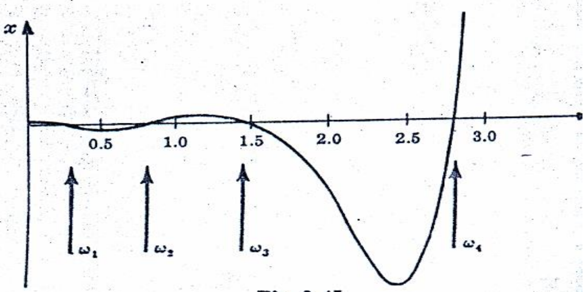

Fig. 3-45

BRANCHED SYSTEM

31. A four spring-mass branched system is shown in Fig. 3-46. If the masses are moving in the vertical direction only, derive the frequency equation of the system.

The equations of motion are given by $\Sigma F = ma$:

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$
$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_3(x_2 - x_3) - k_4(x_2 - x_4)$$
$$m_3 \ddot{x}_3 = -k_3(x_3 - x_2)$$
$$m_4 \ddot{x}_4 = -k_4(x_4 - x_2)$$

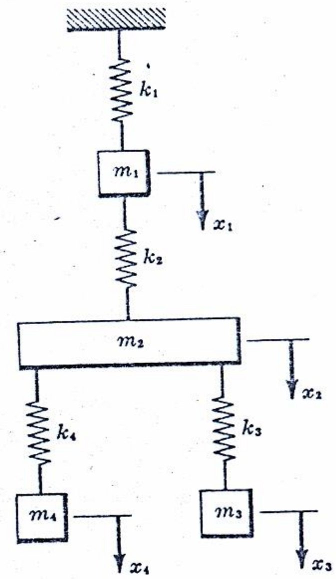

Fig. 3-46

Rearranging,

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$
$$m_2 \ddot{x}_2 + (k_2 + k_3 + k_4)x_2 - k_3 x_3 - k_4 x_4 - k_2 x_1 = 0$$
$$m_3 \ddot{x}_3 + k_3 x_3 - k_3 x_2 = 0$$
$$m_4 \ddot{x}_4 + k_4 x_4 - k_4 x_2 = 0$$

Assume the motion is periodic and is composed of harmonic components of various amplitudes and frequencies. Let

$$x_1 = A \cos(\omega t + \psi), \qquad \ddot{x}_1 = -\omega^2 A \cos(\omega t + \psi)$$
$$x_2 = B \cos(\omega t + \psi), \qquad \ddot{x}_2 = -\omega^2 B \cos(\omega t + \psi)$$
$$x_3 = C \cos(\omega t + \psi), \qquad \ddot{x}_3 = -\omega^2 C \cos(\omega t + \psi)$$
$$x_4 = D \cos(\omega t + \psi), \qquad \ddot{x}_4 = -\omega^2 D \cos(\omega t + \psi)$$

When these relations are substituted and the term $\cos(\omega t + \psi)$ cancelled out, the differential equations of motion become a set of algebraic equations:

$$(k_1 + k_2 - m_1\omega^2)A - k_2 B = 0$$
$$-k_2 A + (k_2 + k_3 + k_4 - m_2\omega^2)B - k_3 C - k_4 D = 0$$
$$-k_3 B + (k_3 - m_3\omega^2)C = 0$$
$$-k_4 B + (k_4 - m_4\omega^2)D = 0$$

from which the frequency equation is obtained by setting the determinant of the coefficients A, B, C, D to zero.

$$\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 & 0 \\ -k_2 & (k_2 + k_3 + k_4 - m_2\omega^2) & -k_3 & -k_4 \\ 0 & -k_3 & (k_3 - m_3\omega^2) & 0 \\ 0 & -k_4 & 0 & (k_4 - m_4\omega^2) \end{vmatrix} = 0$$

Expand the determinant and simplify to obtain

$$\omega^8 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3 + k_4}{m_2} + \frac{k_3}{m_3} + \frac{k_4}{m_4}\right]\omega^6$$
$$+ \left[\frac{k_1k_2 + k_2k_3 + k_3k_1 + k_1k_4 + k_2k_4}{m_1m_2} + \frac{k_2k_3 + k_3k_4}{m_2m_3} + \frac{k_1k_3 + k_2k_3}{m_1m_3} + \frac{k_2k_4 + k_3k_4}{m_2m_4} + \frac{k_1k_4 + k_2k_4}{m_1m_4} + \frac{k_3k_4}{m_3m_4}\right]\omega^4$$
$$- \left[\frac{k_1k_2k_3 + k_2k_3k_4 + k_3k_4k_1}{m_1m_2m_3} + \frac{k_2k_3k_4}{m_2m_3m_4} + \frac{(k_1 + k_2)k_3k_4}{m_3m_4m_1} + \frac{k_4(k_1k_2 + k_2k_3 + k_3k_1)}{m_4m_1m_2}\right]\omega^2$$
$$+ \frac{k_1k_2k_3k_4}{m_1m_2m_3m_4} = 0$$

33. Use the Stodola method to determine the lowest natural frequency of the branched system as shown in Fig. 3-48.

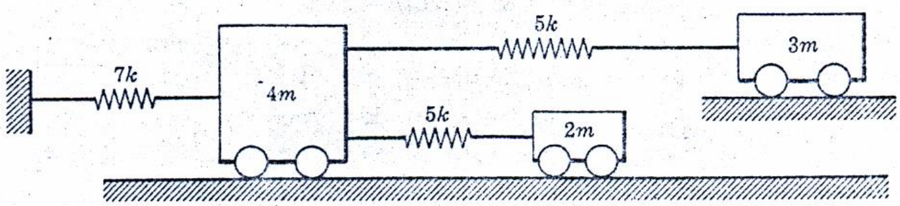

Fig. 3-48

See Problem 26, Page 98, for an explanation of the Stodola method.

	$k_1 = 7k$	$m_1 = 4m$	$k_2 = 5k$	$m_2 = 3m$	$k_3 = 5k$	$m_3 = 2m$
Assumed deflection		1		1		1
Inertia force		$4\omega^2$		$3\omega^2$		$2\omega^2$
Spring force	$9\omega^2$		$3\omega^2$		$2\omega^2$	
Spring deflection	$1.3\omega^2$		$0.6\omega^2$		$0.4\omega^2$	
Calculated deflection		$1.3\omega^2$		$1.9\omega^2$		$1.7\omega^2$
		1		1.46		1.31
Assumed deflection		1		1.4		1.3
Inertia force		$4\omega^2$		$4.2\omega^2$		$2.6\omega^2$
Spring force	$10.8\omega^2$		$4.2\omega^2$		$2.6\omega^2$	
Spring deflection	$1.54\omega^2$		$0.84\omega^2$		$0.52\omega^2$	
Calculated deflection		$1.54\omega^2$		$2.38\omega^2$		$2.06\omega^2$
		1		1.54		1.34
Assumed deflection		1		1.52		1.34
Inertia force		$4\omega^2$		$4.56\omega^2$		$2.68\omega^2$
Spring force	$11.24\omega^2$		$4.56\omega^2$		$2.68\omega^2$	
Spring deflection	$1.61\omega^2$		$0.92\omega^2$		$0.53\omega^2$	
Calculated deflection		$1.61\omega^2$		$2.53\omega^2$		$2.14\omega^2$
		1		1.56		1.32
Assumed deflection		1		1.56		1.32
Inertia force		$4\omega^2$		$4.68\omega^2$		$2.64\omega^2$
Spring force	$11.32\omega^2$		$4.68\omega^2$		$2.64\omega^2$	
Spring deflection	$1.62\omega^2$		$0.93\omega^2$		$0.53\omega^2$	
Calculated deflection		$1.62\omega^2$		$2.55\omega^2$		$2.15\omega^2$
		1		1.57		1.33

The assumed deflection $\begin{bmatrix} 1.00 \\ 1.56 \\ 1.32 \end{bmatrix}$ at this point is quite close to the calculated deflection. Hence the fundamental principal mode of vibration is given by $\begin{bmatrix} 1.00 \\ 1.57 \\ 1.33 \end{bmatrix}$ and the lowest natural frequency is found from

$$(1 + 1.57 + 1.33) = (1.62 + 2.55 + 2.15)\omega^2 \qquad \text{or} \qquad \omega_1 = 0.79\sqrt{k/m} \text{ rad/sec}$$

33. Use the Stodola method to determine the lowest natural frequency of the branched system as shown in Fig. 3-48.

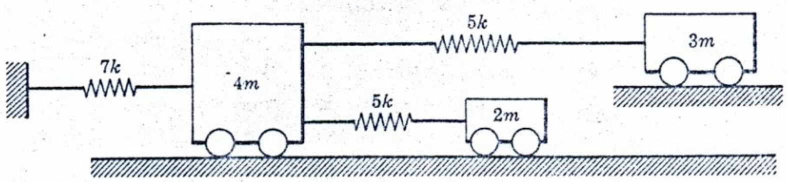

Fig. 3-48

See Problem 26, Page 98, for an explanation of the Stodola method.

	$k_1 = 7k$	$m_1 = 4m$	$k_2 = 5k$	$m_2 = 3m$	$k_3 = 5k$	$m_3 = 2m$
Assumed deflection		1		1		1
Inertia force		$4\omega^2$		$3\omega^2$		$2\omega^2$
Spring force	$9\omega^2$		$3\omega^2$		$2\omega^2$	
Spring deflection	$1.3\omega^2$		$0.6\omega^2$		$0.4\omega^2$	
Calculated deflection		$1.3\omega^2$		$1.9\omega^2$		$1.7\omega^2$
		1		1.46		1.31
Assumed deflection		1		1.4		1.3
Inertia force		$4\omega^2$		$4.2\omega^2$		$2.6\omega^2$
Spring force	$10.8\omega^2$		$4.2\omega^2$		$2.6\omega^2$	
Spring deflection	$1.54\omega^2$		$0.84\omega^2$		$0.52\omega^2$	
Calculated deflection		$1.54\omega^2$		$2.38\omega^2$		$2.06\omega^2$
		1		1.54		1.34
Assumed deflection		1		1.52		1.34
Inertia force		$4\omega^2$		$4.56\omega^2$		$2.68\omega^2$
Spring force	$11.24\omega^2$		$4.56\omega^2$		$2.68\omega^2$	
Spring deflection	$1.61\omega^2$		$0.92\omega^2$		$0.53\omega^2$	
Calculated deflection		$1.61\omega^2$		$2.53\omega^2$		$2.14\omega^2$
		1		1.56		1.32
Assumed deflection		1		1.56		1.32
Inertia force		$4\omega^2$		$4.68\omega^2$		$2.64\omega^2$
Spring force	$11.32\omega^2$		$4.68\omega^2$		$2.64\omega^2$	
Spring deflection	$1.62\omega^2$		$0.93\omega^2$		$0.53\omega^2$	
Calculated deflection		$1.62\omega^2$		$2.55\omega^2$		$2.15\omega^2$
		1		1.57		1.33

The assumed deflection $\begin{bmatrix} 1.00 \\ 1.56 \\ 1.32 \end{bmatrix}$ at this point is quite close to the calculated deflection. Hence the fundamental principal mode of vibration is given by $\begin{bmatrix} 1.00 \\ 1.57 \\ 1.33 \end{bmatrix}$ and the lowest natural frequency is found from

$$(1 + 1.57 + 1.33) = (1.62 + 2.55 + 2.15)\omega^2 \qquad \text{or} \qquad \omega_1 = 0.79\sqrt{k/m} \text{ rad/sec}$$

THE MECHANICAL IMPEDANCE METHOD

34. Use the Mechanical Impedance method to find the frequency equation of the spring-mass system as shown in Fig. 3-49.

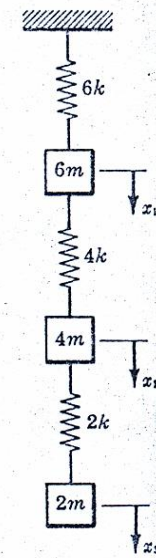

Fig. 3-49

The mechanical impedances are k and $-m\omega^2$ for spring and mass.

For junction x_1, this becomes

$$(6k + 4k - 6m\omega^2)x_1$$

where the slippage term is $4kx_2$. Since there is no force acting on junction x_1, one equation will be obtained as

$$(6k + 4k - 6m\omega^2)x_1 - 4kx_2 = 0$$

Similarly, for junction x_2 the equation is

$$(4k + 2k - 4m\omega^2)x_2 - 4kx_1 - 2kx_3 = 0$$

where $4kx_1$ and $2kx_3$ are the slippage terms.

For junction x_3, this is given by

$$(2k - 2m\omega^2)x_3 - 2kx_2 = 0$$

Rearrange the equations to get

$$(10k - 6m\omega^2)x_1 - 4kx_2 = 0$$
$$-4kx_1 + (6k - 4m\omega^2)x_2 - 2kx_3 = 0$$
$$-2kx_2 + (2k - 2m\omega^2)x_3 = 0$$

Hence the frequency equation is given by

$$\begin{vmatrix} (10k - 6m\omega^2) & -4k & 0 \\ -4k & (6k - 4m\omega^2) & -2k \\ 0 & -2k & (2k - 2m\omega^2) \end{vmatrix} = 0$$

35. Use the Mechanical Impedance method to determine the steady state vibrations of the masses of the system as shown in Fig. 3-50. Let $k_1 = k_2 = k_3 = k_4 = 1$ lb/in, $c_1 = c_2 = c_3 = c_4 = 1$ sec-lb/in, $m_1 = m_2 = m_3 = 1$ lb-sec^2/in, and $\omega = 1$ rad/sec.

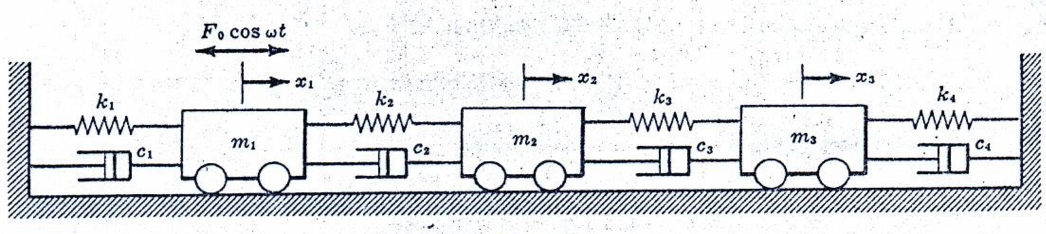

Fig. 3-50

Writing the impedance for junction x_1 and its amplitude, we obtain

$$(k_1 + ic_1\omega - m_1\omega^2 + k_2 + ic_2\omega)x_1$$

and the slippage terms for the junction x_1 are $k_2x_2 + ic_2\omega x_2$; hence the first equation is given by

$$(k_1 + k_2 + ic_1\omega + ic_2\omega - m_1\omega^2)x_1 - k_2x_2 - ic_2\omega x_2 = F_0$$

Similarly, the equations for junctions x_2 and x_3 are

$$(k_2 + k_3 + ic_2\omega + ic_3\omega - m_2\omega^2)x_2 - k_2x_1 - ic_2\omega x_1 - k_3x_3 - ic_3\omega x_3 = 0$$
$$(k_3 + k_4 + ic_3\omega + ic_4\omega - m_3\omega^2)x_3 - k_3x_2 - ic_3x_2\omega = 0$$

Substituting the given values into the equations of motion, we get

$$(1 + 2i)x_1 - (1 + i)x_2 = F_0$$
$$-(1 + i)x_1 + (1 + 2i)x_2 - (1 + i)x_3 = 0$$
$$-(1 + i)x_2 + (1 + 2i)x_3 = 0$$

Solving by Cramer's rule,

$$x_1 = \frac{\begin{vmatrix} F_0 & -(1+i) & 0 \\ 0 & (1+2i) & -(1+i) \\ 0 & -(1+i) & (1+2i) \end{vmatrix}}{\begin{vmatrix} (1+2i) & -(1+i) & 0 \\ -(1+i) & (1+2i) & -(1+i) \\ 0 & -(1+i) & (1+2i) \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} (1+2i) & F_0 & 0 \\ -(1+i) & 0 & -(1+i) \\ 0 & 0 & (1+2i) \end{vmatrix}}{\begin{vmatrix} (1+2i) & -(1+i) & 0 \\ -(1+i) & (1+2i) & -(1+i) \\ 0 & -(1+i) & (1+2i) \end{vmatrix}}$$

$$x_3 = \frac{\begin{vmatrix} (1+2i) & -(1+i) & F_0 \\ -(1+i) & (1+2i) & 0 \\ 0 & -(1+i) & 0 \end{vmatrix}}{\begin{vmatrix} (1+2i) & -(1+i) & 0 \\ -(1+i) & (1+2i) & -(1+i) \\ 0 & -(1+i) & (1+2i) \end{vmatrix}}$$

Expand the determinants to obtain

$$x_1 = \frac{F_0\,(1+2i)(1+2i) - F_0\,(1+i)^2}{(1+2i)^3 - (1+2i)(1+i)^2 - (1+2i)(1+i)^2}$$

$$x_2 = \frac{F_0\,(1+i)(1+2i)}{(1+2i)^3 - (1+2i)(1+i)^2 - (1+2i)(1+i)^2}$$

$$x_3 = \frac{F_0\,(1+i)^2}{(1+2i)^3 - (1+2i)(1+i)^2 - (1+2i)(1+i)^2}$$

The equation for x_1 simplifies to

$$x_1 = \frac{F_0\,(3-2i)}{(3+6i)} = \frac{13F_0}{(-3+24i)}$$

Therefore the numerical value of the amplitude of x_1 is

$$\frac{13F_0}{\sqrt{9+24^2}} = 0.54F_0 \qquad \text{and the phase angle } \phi = \tan^{-1}(24/-3) = -82.9^\circ$$

Thus the steady state response of the mass m_1 is

$$x_1(t) = 0.54F_0 \cos(\omega t - 82.9^\circ)$$

Similar expressions can be found for masses m_2 and m_3 as

$$x_2(t) = 0.47F_0 \cos(\omega t - 45^\circ)$$
$$x_3(t) = 0.29F_0 \cos(\omega t - 26.7^\circ)$$

THE ORTHOGONALITY PRINCIPLE

36. Show that the orthogonality principle holds for Problem 8, Page 80.

For three-degree-of-freedom systems, the orthogonality principle can be written as

$$m_1A_1A_2 + m_2B_1B_2 + m_3C_1C_2 = 0$$
$$m_1A_2A_3 + m_2B_2B_3 + m_3C_2C_3 = 0$$
$$m_1A_3A_1 + m_2B_3B_1 + m_3C_3C_1 = 0$$

where m's are the masses; A's, B's and C's are the amplitudes of vibration of the system.

The general motion of the problem is

$$x_1(t) = \tfrac{1}{3} - \tfrac{1}{2}\sin(t + \pi/2) + \tfrac{1}{6}\sin(\sqrt{3}\,t + \pi/2)$$
$$x_2(t) = \tfrac{1}{3} - \tfrac{1}{3}\sin(\sqrt{3}\,t + \pi/2)$$
$$x_3(t) = \tfrac{1}{3} + \tfrac{1}{2}\sin(t + \pi/2) + \tfrac{1}{6}\sin(\sqrt{3}\,t + \pi/2)$$

Substituting the corresponding amplitudes of vibration into the equations for the principle of orthogonality, we obtain

$$m(\tfrac{1}{3})(-\tfrac{1}{2}) + m(\tfrac{1}{3})(0) + m(\tfrac{1}{3})(\tfrac{1}{2}) = 0$$
$$m(-\tfrac{1}{2})(\tfrac{1}{6}) + m(0)(\tfrac{1}{3}) + m(\tfrac{1}{2})(\tfrac{1}{6}) = 0$$
$$m(\tfrac{1}{6})(\tfrac{1}{3}) + m(-\tfrac{1}{3})(\tfrac{1}{3}) + m(\tfrac{1}{6})(\tfrac{1}{3}) = 0$$

Hence the orthogonality principle is completely satisfied.

Supplementary Problems

37. Derive the equations of motion of the system as shown in Fig. 3-51 below. The connecting rods are weightless and restrict motion to the plane of the paper.

Ans. $4m\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$
$4m\ddot{\theta}_2 + 2k\theta_2 - k\theta_3 - k\theta_1 = 0$
$4m\ddot{\theta}_3 + 2k\theta_3 - k\theta_2 = 0$

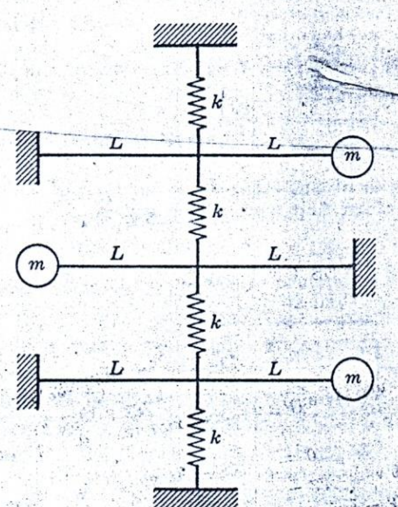

Fig. 3-51

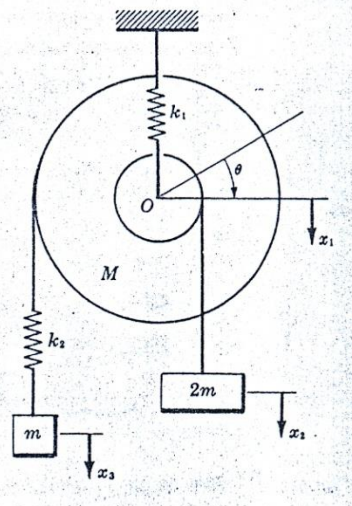

Fig. 3-52

38. The circular homogeneous cylinder of total mass M and radius $2a$ is suspended by a spring of stiffness k_1, and is free to rotate with respect to its center of mass O as shown in Fig. 3-52 above. Derive the equations of motion.

Ans. $3M\ddot{x}_1 + (k_1 + 9k_2)x_1 - 2M\ddot{x}_2 - 6k_2x_2 - 3k_2x_3 = 0$
$(2M + 2m)\ddot{x}_2 + 4k_2x_2 + 2k_2x_3 - 2M\ddot{x}_1 - 6k_2x_1 = 0$
$m\ddot{x}_3 + k_2x_3 - 3k_2x_1 + 2k_2x_2 = 0$

39. Calculate the natural frequencies of the system as shown in Fig. 3-53.

Ans. $\omega_1 = 0.39\sqrt{k/m}$, $\omega_2 = 1.47\sqrt{k/m}$,
$\omega_3 = 2.36\sqrt{k/m}$ rad/sec

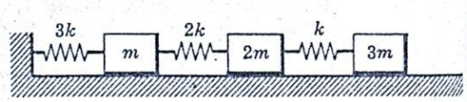

Fig. 3-53